Lecture 10

Partial differential equations

Laplace equation: $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ (can be also written as $\nabla^2 u = 0$ or $\Delta u = 0$)

Solutions are called harmonic functions (graphs are soap films). Some simple solutions are $1, x, y$, $x^2 - y^2$ (the saddle shape we saw earlier), $2xy$ ....

* Wave equation: $\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}$

Solution describes a wave form (x here is the spatial position and t is the time).

* heat equation (or diffusion equation)

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}$$ u here is the temperature

$\Rightarrow$ this equation modelizes the thermal diffusion.

* Navier - Stokes equations

$$\frac{\partial \underline{u}}{\partial t} + \underline{u}.\nabla \underline{u} = -\nabla p + \frac{1}{Re} \Delta \underline{u}$$

here u is the velocity field (it is a vector) and p is the pressure field (it is a scalar).

Re is a non-dimensional number it caracterizes the ratio between the viscous and inertial effects. it is the Reynolds number

The proof of the existence and smoothness of the Navier-Stokes equation is still an open question. There is a 1million dolar reward to who proves it (millenium problem)

* Example of solution to the heat equation *

Le $u(x,t) = f(t)\, e^{-x^2/4t}$, find $f(t)$ so that $u(x,t)$ satisfies the heat equation $\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}$

$$\frac{\partial u}{\partial t} = f'(t)\, e^{-x^2/4t} + f(t)\left(\frac{x^2}{4t^2}\right) e^{-x^2/4t}$$

$$\frac{\partial u}{\partial x} = \left(-\frac{2x}{4t} e^{-x^2/4t}\right) f(t) \Rightarrow \frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right)$$

③

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial}{\partial x}\left(f(t)\left(-\frac{2x}{4t}e^{-x^2/4t}\right)\right)$$

$$\frac{\partial^2 u}{\partial x^2} = f(t)\left(-\frac{2}{4t}e^{-x^2/4t} + \left(-\frac{2x}{4t}\right)\left(-\frac{2x}{4t}\right)e^{-x^2/4t}\right)$$

$$\frac{\partial^2 u}{\partial x^2} = f(t)\left(-\frac{1}{2t}e^{-x^2/4t} + \frac{1x^2}{4t^2}e^{-x^2/4t}\right)$$

=> match both side and extract $f(t)$.

$$f'(t)e^{-x^2/4t} + \cancel{f(t)\frac{x^2}{4t^2}e^{-x^2/4t}} = f(t)\left(\frac{-1}{2t}\right)e^{-x^2/4t} + \cancel{f(t)e^{-x^2/4t}}\frac{x^2}{4t^2}$$

$$f'(t)e^{-x^2/4t} = f(t)\left(\frac{-1}{2t}\right)e^{-x^2/4t}$$

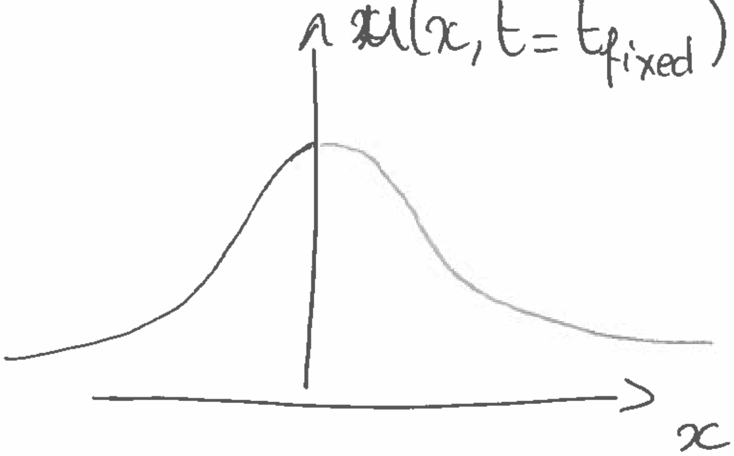
$$\Rightarrow \frac{f'(t)}{f(t)} = \frac{-1}{2t} \Rightarrow \left(\log(f(t))\right)' = \frac{-1}{2t}$$

$$\log f(t) = \frac{-1}{2}\log t + C_1 \quad (C_1 \text{ is a constant}).$$

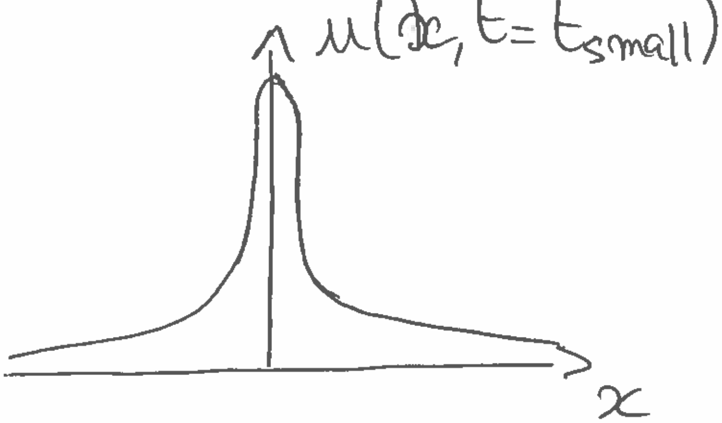
$$\log f(t) = \log\left(\frac{1}{\sqrt{t}}\right) + \log(C_2) \quad (\log C_2 = C_1)$$

$$\log f(t) = \log\left(\frac{C_2}{\sqrt{t}}\right) \Rightarrow \boxed{f(t) = \frac{C}{\sqrt{t}} \Rightarrow u(x,t) = \frac{C}{\sqrt{t}}e^{-x^2/4t}}$$

The equation $u(x,t) = \frac{1}{\sqrt{t}} e^{-x^2/4t}$ (we chose $C=1$) satisfies the heat equation: for fixed time $t>0$, $u(x,t)$ is a "bell curve"



=> for t small, the bell is very sharp.



=> for t larger, curve flattens: the heat diffuses.