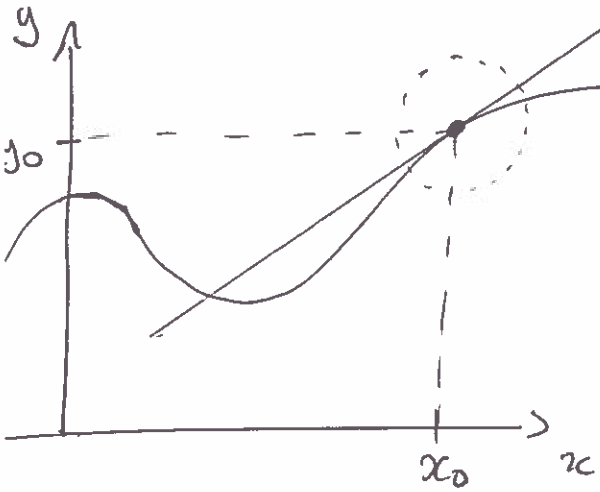
Linear approximations and Tangent Planes | Lecture 10

one variable:

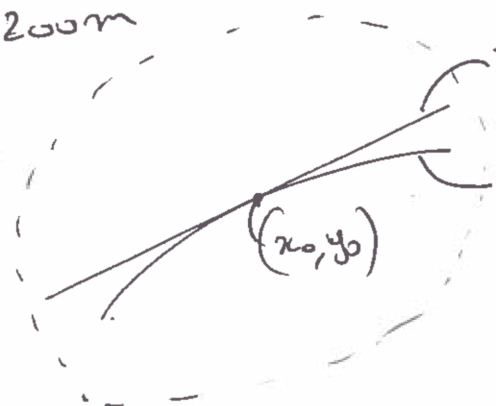
$$y_t = y_0 + f'(x_0)(x - x_0)$$

$$y_0 = f(x_0)$$



this is the equation of the line passing through (x_0, y_0) having slope $f'(x_0)$.

zoom



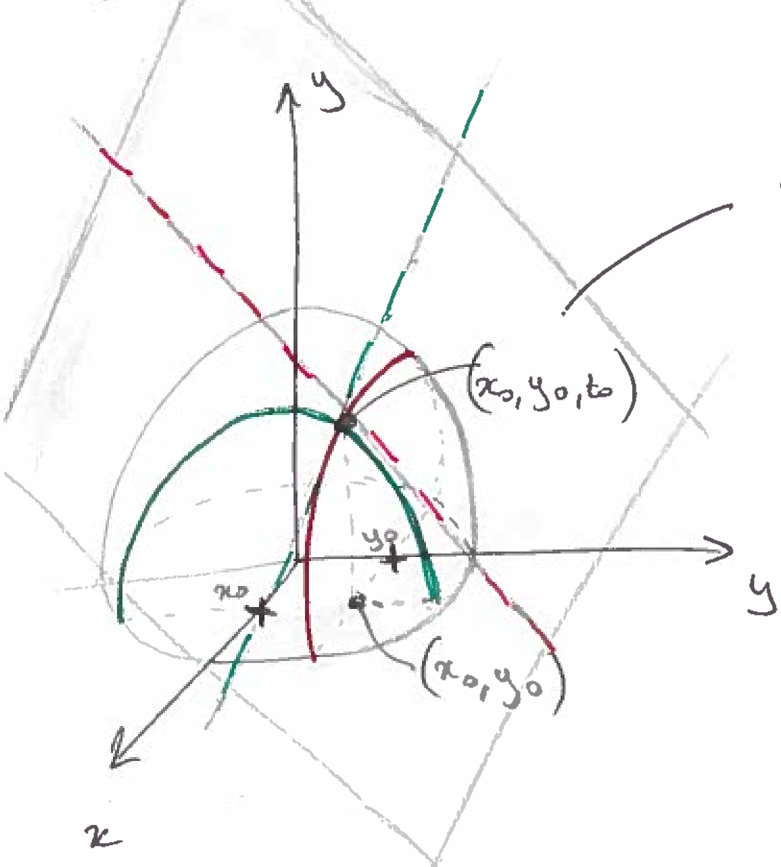
The linear approximation $f(x) \approx f(x_0) + f'(x_0)(x - x_0)$.

This approximation gets better the closer x get to x_0

$\Rightarrow$ what is the best linear approximation to $z = f(x,y)$ at a point (x_0, y_0)?

$\Rightarrow$ it is given by the tangent plane

$z = f(x,y) \qquad z_0 = f(x_0, y_0)$

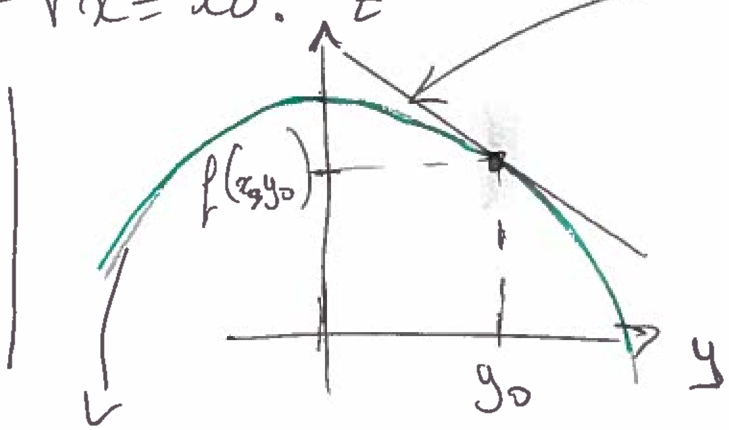


plane tangent to graph at (x_0, y_0, z_0) with $z_0 = f(x_0, y_0)$

$\Rightarrow$ it is the plane containing the two lines tangent to the trace curves.

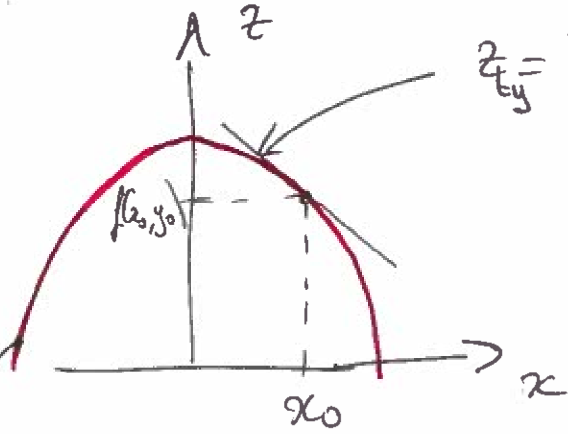
plane at $x = x_0$:

$z_{tx} = z_0 + \dfrac{\partial f}{\partial y}(x_0, y_0)(y - y_0)$



$z = f(x_0, y)$

at plane $y = y_0$



$$z_{ty} = z_0 + \frac{\partial f}{\partial x}(x_0,y_0)(x-x_0)$$

$$z = f(x, y_0)$$

$\Rightarrow$ Tangent plane has the form (with $\langle A, B, C\rangle$ the normal vector)

$$(*) \quad A(x-x_0) + B(y-y_0) + C(z-z_0) = 0$$

$\hookrightarrow \dfrac{(*)}{C}$ ($C \neq 0$ since the plane isn't vertical)

it comes

$$z - z_0 = -\frac{A}{C}(x-x_0) - \frac{B}{C}(y-y_0)$$

if we set $\cancel{x = x_0}$, we should get $z = z_{tx} = z_0 + \dfrac{\partial f}{\partial y}(x_0,y_0)(y-y_0)$

$$\left.\begin{aligned} &\Rightarrow z = z_0 - \frac{B}{C}(y-y_0) \\ &z_{tx} = z_0 + \frac{\partial f}{\partial y}(x_0,y_0)(y-y_0) \end{aligned}\right\} \quad -\frac{B}{C}\cancel{(y-y_0)} = \frac{\partial f}{\partial y}(x_0,y_0)\cancel{(y-y_0)}$$

similarly, if we set $y = y_0$, we should get

$$\left.\begin{aligned} z &= z_0 + \frac{\partial f}{\partial x}(x_0, y_0)(x - x_0) \\ z_{t_y} &= z_0 - \frac{A}{C}(x - x_0) \end{aligned}\right\} \quad \frac{\partial f(x_0,y_0)}{\partial x} = -\frac{A}{C}$$

it comes that the equation of the tangent plane is

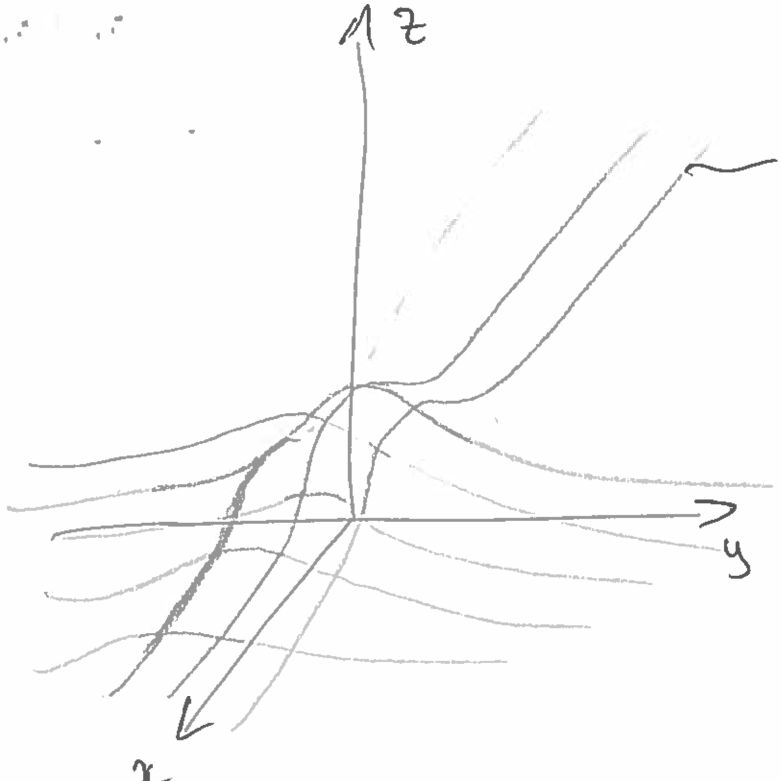
$$\boxed{z = f(x_0, y_0) + \frac{\partial f}{\partial x}(x_0, y_0)(x - x_0) + \frac{\partial f}{\partial y}(x_0, y_0)(y - y_0)}$$

can be also written as

$$z = z_0 + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

⚠ note similarity to $y = f(x_0) + f'(x_0)(x - x_0)$

<u>example</u>: Find the equation of the tangent plane to $z = e^{-x^2 - y^2}$ at $(x_0, y_0) = (1, 1)$ and at $(x_0, y_0) = (0, 0)$



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$\Rightarrow$ for $(x_0, y_0) = (1,1)$ $\quad z_0 = e^{-2} = \dfrac{1}{e^2}$

$\dfrac{\partial z}{\partial x} = -2x\, e^{-x^2-y^2} \Rightarrow \dfrac{\partial z}{\partial x}(1,1) = -2\, e^{-2}$

$\dfrac{\partial z}{\partial y} = -2y\, e^{-x^2-y^2} \Rightarrow \dfrac{\partial z}{\partial y}(1,1) = -2e^{-2}$

$z = e^{-2} - 2e^{-2}(x-1) + -2e^{-2}(y-1)$

$z = e^{-2}\left(1 - 2(x-1) - 2(y-1)\right)$

$z = e^{-2}\left[5 - 2x - 2y\right]$ is the equation of the plane at $x_0=1, y_0=1$