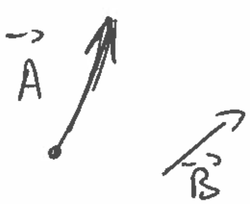
Lecture notes 2

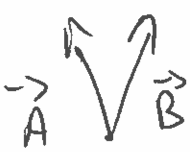
* Vectors

Vectors are quantities with both magnitude and direction. We indicate a vector by an arrow (in $\mathbb{R}^2$ or $\mathbb{R}^3$)

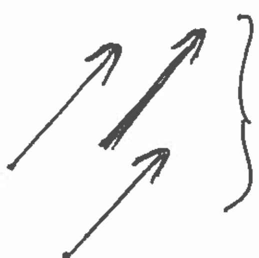
magnitude is indicated by lenght of arrow



$\vec{A}$ and $\vec{B}$ have different magnitude and direction



same magnitude, different direction.



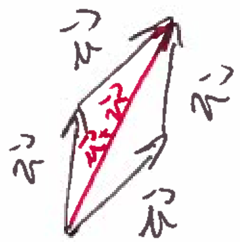
all the same vector

Many quantities in math and real life have both direction and magnitude: velocity, force, gravity acceleration, magnetic field...

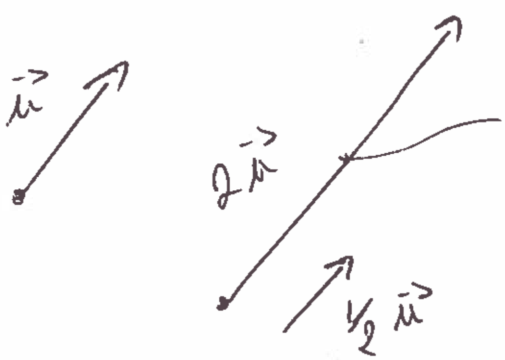
• **definition** : Addition is done by placing vectors end to end

$\vec{u} + \vec{v}$ is the vector :

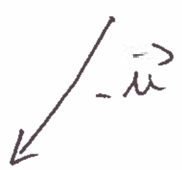
note $\vec{u}+\vec{v} = \vec{v} + \vec{u}$



• **definition** we can multiply a vector by a scalar



same direction, 2 magnitude



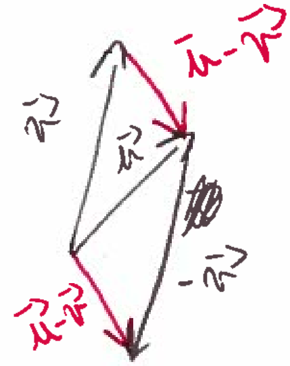
opposite direction than $\vec{u}$

$\vec{u} - \vec{u} = \vec{0}$

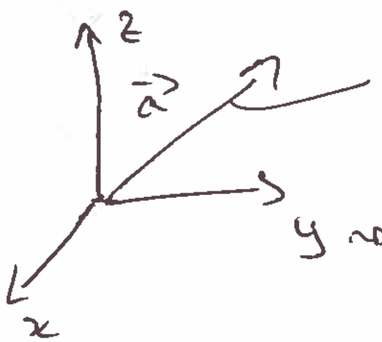
• The zero vector $\vec{0}$ is the only vector without direction, it is the only vector of magnitude 0.

(if the wind isn't blowing, it doesn't make sense to talk about which direction it isn't blowing)

Note $\vec{u} - \vec{v} = \vec{u} + (-\vec{v})$



$\Rightarrow$ We did all these vector operations without using a coordinate system. The vector from the origin $(0,0,0)$ to the point (a_1, a_2, a_3) is called $\langle a_1, a_2, a_3 \rangle$.

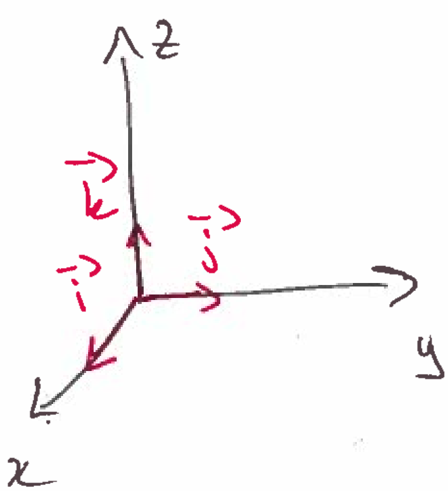


This vector $\vec{a} = \langle a_1, a_2, a_3 \rangle$. a_1, a_2 and a_3 are the components of $\vec{a}$.

in 2D:

$\vec{a} = \langle a_1, a_2 \rangle$

$\Rightarrow$ We will use the $\vec{i}, \vec{j}, \vec{k}$ notation: ie the standard basis vectors



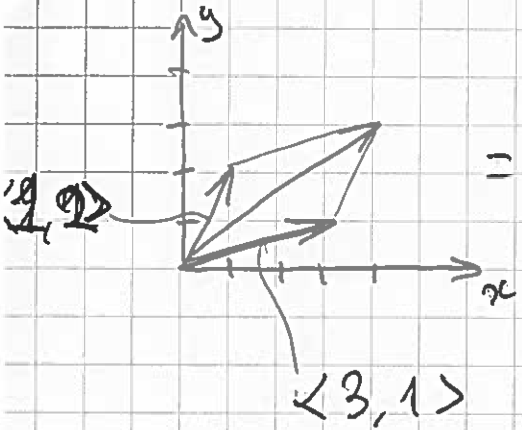
$\vec{i} = \langle 1, 0, 0 \rangle$

$\vec{j} = \langle 0, 1, 0 \rangle$

$\vec{k} = \langle 0, 0, 1 \rangle$

Then $\vec{a} = \langle a_1, a_2, a_3 \rangle = a_1 \vec{i} + a_2 \vec{j} + a_3 \vec{k}$

* Vector addition is equivalent to adding components:



$$\langle 1,2\rangle + \langle 3,1\rangle = \langle 4,3\rangle$$
$$= 1\vec{i} + 2\vec{j} + 3\vec{i} + 1\vec{j} = 4\vec{i} + 3\vec{j}$$

* similarly $c\vec{a} = \langle ca_1, ca_2, ca_3\rangle$

$$c\vec{a} = ca_1\vec{i} + ca_2\vec{j} + ca_3\vec{k}$$

* magnitude $|\vec{a}|$ with $\vec{a} = \langle a_1, a_2, a_3\rangle$

$$|\vec{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}$$

* operations: on scalar (numbers) and vectors:

we have seen:

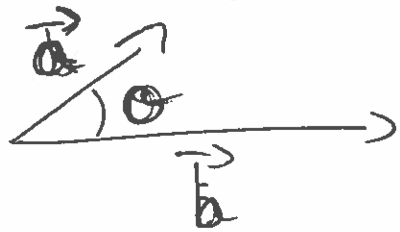
-> add two vectors and get a vector

-> multiply a vector by a scalar and get a vector.

* will see now Dot product of two vectors: get a scalar (number)

* Definitions of the dot product

geometric



$\vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|\cos\theta$

Algebraic

$\vec{a} = \langle a_1, a_2, a_3 \rangle$

$\vec{b} = \langle b_1, b_2, b_3 \rangle$

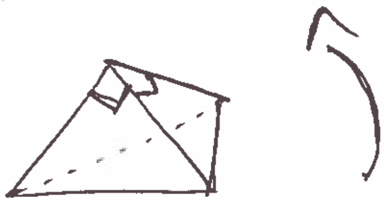
$\vec{a}\cdot\vec{b} = a_1b_1 + a_2b_2 + a_3b_3$

⚠ These two definitions are the same.

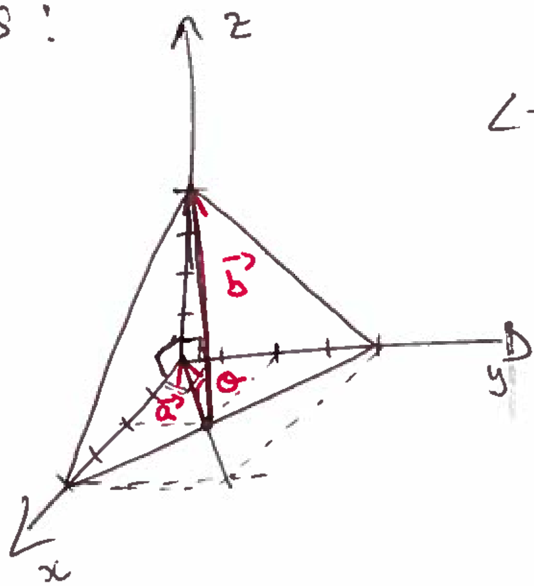
⇝ notes: $\vec{a}\cdot\vec{b} = 0$ means that $\vec{a}$ is perpendicular to $\vec{b}$, $\theta = \pi/2$ or $-\pi/2$. we note $\vec{a} \perp \vec{b}$.

⇝ $|\vec{a}|^2 = \vec{a}\cdot\vec{a}$

<u>example</u>: A pyramid is built from 3 $45°-45°-90°$ triangles as side and an equilateral triangle as base. What angle does the side make with the base?



$\rightsquigarrow$ Turn the pyramid so that the top corner is the origin with the sides in the coordinate planes:



we want the angle θ between the vector $\vec{a}$ and $\vec{b}$.

$\vec{a}$ is $\langle -\frac{1}{2}, -\frac{1}{2}, 0 \rangle$

$\vec{b}$ is $\vec{a} + \vec{k}$

$\vec{b} = \langle -\frac{1}{2}, -\frac{1}{2}, 1 \rangle$

$$\vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|\cos\theta = \sqrt{\tfrac{1}{4}+\tfrac{1}{4}}\sqrt{\tfrac{1}{4}+\tfrac{1}{4}+1}\,\cos\theta$$

$$\tfrac{1}{4}+\tfrac{1}{4} = \sqrt{\tfrac{1}{2}}\sqrt{\tfrac{3}{2}}\cos\theta \Rightarrow \cos\theta = \frac{1}{\sqrt{3}} \Rightarrow \theta = \cos^{-1}\left(\frac{1}{\sqrt{3}}\right)$$

$$\Rightarrow \boxed{\theta \approx 55^\circ}$$

* some properties

Many algebraic relations you know for scalars hold for dot products.

$$\vec{a}\cdot(\vec{b}+\vec{c}) = \vec{a}\cdot\vec{b}+\vec{a}\cdot\vec{c}$$

if c is a scalar then

$$c(\vec{a}\cdot\vec{b}) = (c\vec{a})\cdot\vec{b} = \vec{a}\cdot(c\vec{b})$$

⚠ Be careful to not mix types

=> example of something that doesn't make sens

$$\underbrace{(\vec{a}\cdot\vec{b})}_{\text{scalar}}+\underbrace{\vec{d}}_{\text{vector}}$$

you can't add a scalar to a vector