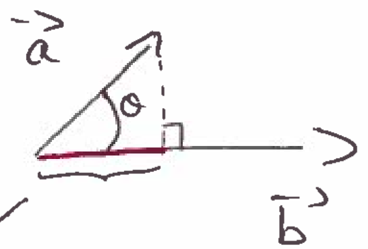
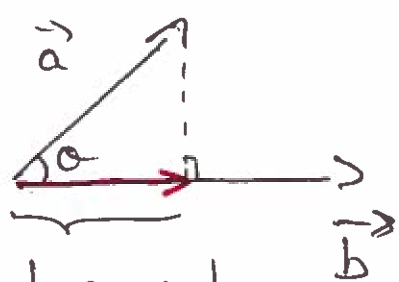
Projections : Lecture 5

One nice way to think about dot product is in terms of projections:



this distance (in red) = scalar projection of $\vec{a}$ on $\vec{b}$ $= \cos\theta\,|\vec{a}| = \dfrac{\vec{a}\cdot\vec{b}}{|\vec{b}|}$



vector projection

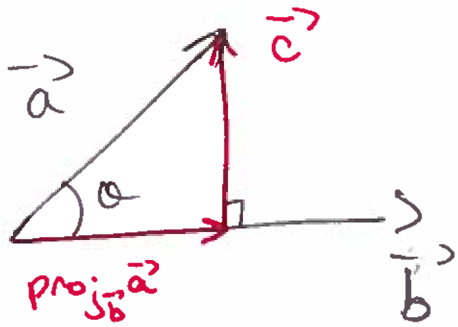
$$\mathrm{proj}_{\vec{b}}\vec{a} = \underbrace{\frac{(\vec{a}\cdot\vec{b})}{|\vec{b}|}}_{\text{magnitude of } \mathrm{proj}_{\vec{b}}\vec{a}} \frac{\vec{b}}{|\vec{b}|}$$

unit vector in the direction of $\vec{b}$

$$\mathrm{proj}_{\vec{b}}\vec{a} = \frac{(\vec{a}\cdot\vec{b})}{|\vec{b}|^2}\,\vec{b}$$

"orthogonal projection" or "vector rejection":

defined by $\vec{c} = \vec{a} - \text{proj}_{\vec{b}}\vec{a}$



$\vec{c}$ is orthogonal to $\vec{b}$; $\text{proj}_{\vec{b}}\vec{a} + \vec{c} = \vec{a}$

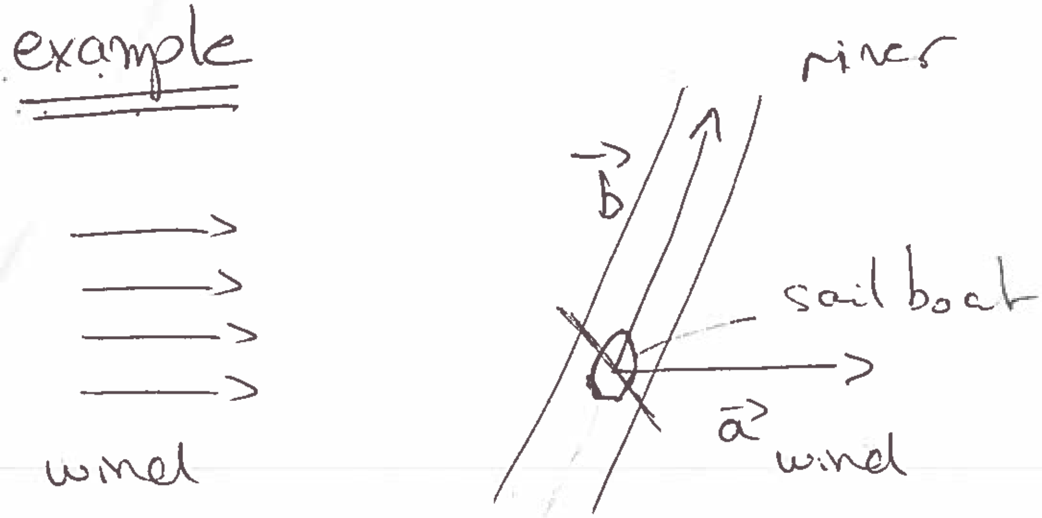
you can allways decompose a vector into a parallel one and an orthogonal one (with respect to another vector).

<u>Note</u>: $\left(\vec{a} - \text{proj}_{\vec{b}}\vec{a}\right) \cdot \vec{b} = \left(\vec{a} - \dfrac{(\vec{a}\cdot\vec{b})\vec{b}}{|\vec{b}|^2}\right) \cdot \vec{b}$

$= \vec{a}\cdot\vec{b} - \dfrac{(\vec{a}\cdot\vec{b})\,\vec{b}\cdot\vec{b}}{|\vec{b}|^2} = 0$ (we used $\vec{b}\cdot\vec{b} = |\vec{b}|^2$)

Proof that $\vec{c}$ is orthogonal to $\vec{b}$.

example



only the projection of the wind vector $\vec{a}$ acts on the boat.

force of wind on sailbot $proj_{\vec{b}}\vec{a} = \frac{\vec{a} \cdot \vec{b}}{|\vec{b}|}$

* ~~[illegible]~~ properties here

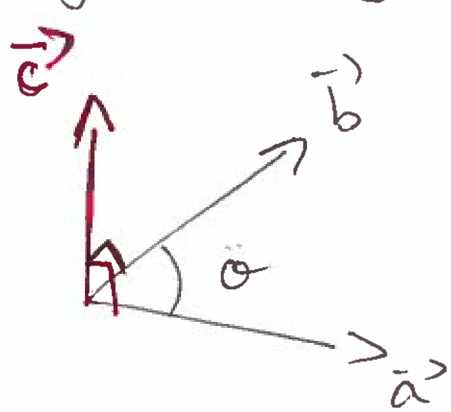
cross product

$\vec{a} \times \vec{b} = \vec{c}$

dot product: 2 vectors $\to$ scalar
cross product: 2 vectors $\to$ vector

* definitions

geometrically



$\vec{a} \times \vec{b} = \vec{c}$ is perpendicular to $\vec{a}$ and $\vec{b}$ (use right hand rule)

and $|\vec{c}| = |\vec{a}||\vec{b}| \sin\theta$

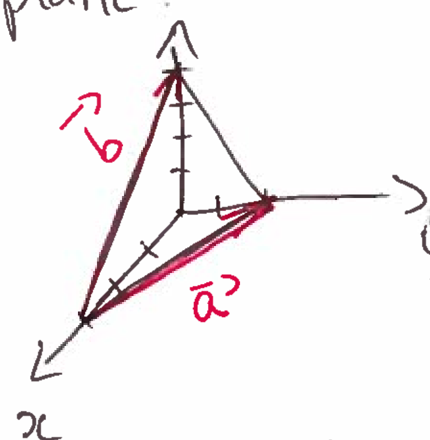
Algebraically

$\langle a_1, a_2, a_3\rangle \times \langle b_1, b_2, b_3\rangle = \langle a_2 b_3 - a_3 b_2, a_3 b_1 - a_1 b_3, a_1 b_2 - b_1 a_2\rangle$

mnemonic:

$$\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = (a_2 b_3 - b_2 a_3)\vec{i} + (a_3 b_1 - a_1 b_3)\vec{j} + (a_1 b_2 - b_1 a_2)\vec{k}$$

example: Consider the plane passing through the points $(3,0,0)$, $(0,2,0)$ and $(0,0,4)$ and find a unit vector (magnitude 1) normal (perpendicular) to the plane.



$\vec{a} = -3\vec{i} + 2\vec{j}$

$\vec{b} = -3\vec{i} + 4\vec{k}$

$\rightsquigarrow \vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -3 & 2 & 0 \\ -3 & 0 & 4 \end{vmatrix} = 8\vec{i} + 12\vec{j} + 6\vec{k}$

$\rightsquigarrow$ and make it a unit vector by dividing it by its length (magnitude):

$$\frac{\vec{a} \times \vec{b}}{|\vec{a} \times \vec{b}|} = \frac{1}{\sqrt{8^2 + 12^2 + 6^2}} \langle 8, 12, 6\rangle = \frac{1}{2\sqrt{61}} \langle 8, 12, 6\rangle = \left\langle \frac{4}{\sqrt{61}}, \frac{6}{\sqrt{61}}, \frac{3}{\sqrt{61}}\right\rangle$$

$= 244 = 4 \cdot 61$

⚠ Cross product is <u>not</u> commutative

$$\vec{a} \times \vec{b} = -\vec{b} \times \vec{a} \quad \text{(right hand rule)}$$

Some familiar properties hold

$$\vec{a} \times (\vec{b} + \vec{c}) = \vec{a} \times \vec{b} + \vec{a} \times \vec{b} \quad \text{(but careful to preserve order)}$$

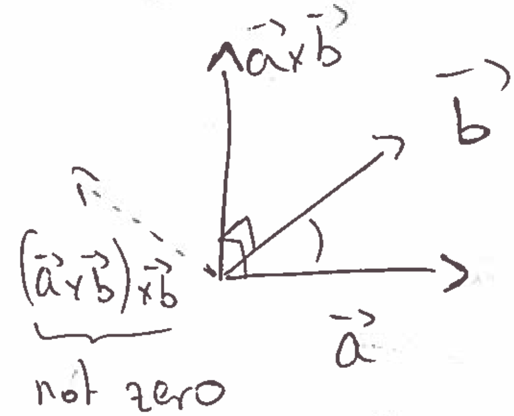
also $\vec{a} \cdot (\vec{b} \times \vec{c}) = (\vec{a} \times \vec{b}) \cdot \vec{c}$

$\rightsquigarrow$ but why not $(\vec{a} \cdot \vec{b}) \times \vec{c}$?)

but $\vec{a} \times (\vec{b} \times \vec{c}) \neq (\vec{a} \times \vec{b}) \times \vec{c}$

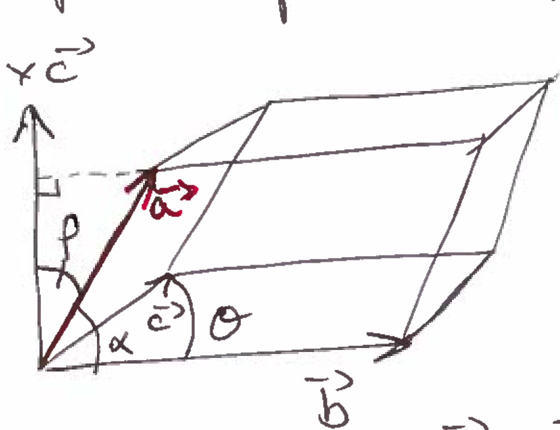
why? it better holds for $\vec{b} = \vec{c}$ if it holds in general.

$$\rightsquigarrow \vec{a} \times \underbrace{(\vec{b} \times \vec{b})}_{\vec{0}} \stackrel{?}{=} (\vec{a} \times \vec{b}) \times \vec{b}$$



in fact $\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \cdot \vec{c})\vec{b} - (\vec{a} \cdot \vec{b})\vec{c}$

The scalar $\vec{a} \cdot (\vec{b} \times \vec{c})$ is the volume (up to the sign) of the parallelepiped



Volume is $= a_1\vec{i} + a_2\vec{j} + a_3\vec{k} \cdot \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$

$$= a_1(b_2c_3 - c_2b_3) - a_2(b_1c_3 - c_1b_3) + a_3(b_1c_2 - c_1b_2)$$

or geometrically

Volume is the surface area of the base × the altitude

⇝ the altitude is the scalar projection of vector $\vec{a}$ on $\vec{b} \times \vec{c}$

$$\leadsto \text{Volume} = \underbrace{|\vec{b} \times \vec{c}|}_{\text{area of base}} \cdot \underbrace{|a| \cos\varphi}_{\text{altitude}}$$

$$\text{Volume} = (\vec{b} \times \vec{c}) \cdot \vec{a}$$

note that:

$\vec{a} \cdot (\vec{b} \times \vec{c}) = -\vec{a} \cdot (\vec{c} \times \vec{b})$

$\vec{a} \cdot (\vec{b} \times \vec{c}) = -\vec{b} \cdot (\vec{a} \times \vec{c})$

$\vec{a} \cdot (\vec{b} \times \vec{c}) = -\vec{c} \cdot (\vec{b} \times \vec{a})$

but

$\vec{a} \cdot (\vec{b} \times \vec{c}) = \vec{b} \cdot (\vec{c} \times \vec{a})$

$= \vec{c} \cdot (\vec{a} \times \vec{b})$

(circular permutation)