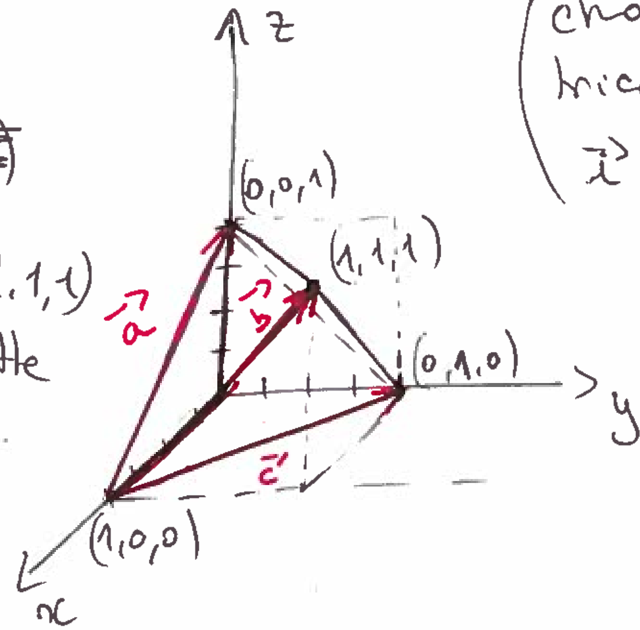
Problem What is the angle between the faces of a regular tetrahedron? ①

Lecture 41

(choose base to be triangle determined by $\vec{i}, \vec{j}, \vec{k}$.)

The apex is $(1,1,1)$ it is $\sqrt{2}$ from the other 3 vertices.



$\vec{a} = \langle -1, 0, 1\rangle$

$\vec{b} = \langle 0, 1, 1\rangle$

$\vec{c} = \langle -1, 1, 0\rangle$

angle between faces is the same as the angle between normal vectors.

normal to face determined by $\vec{a}, \vec{b}$ is

$$\vec{n}_1 = \vec{b}\times\vec{a} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0 & 1 & 1 \\ -1 & 0 & 1 \end{vmatrix} = \langle 1, -1, 1\rangle$$

normal to face determined by $\vec{b}, \vec{c}$

$$\vec{n}_2 = \vec{b}\times\vec{c} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0 & 1 & 1 \\ -1 & 1 & 0 \end{vmatrix} = \langle -1, -1, 1\rangle$$

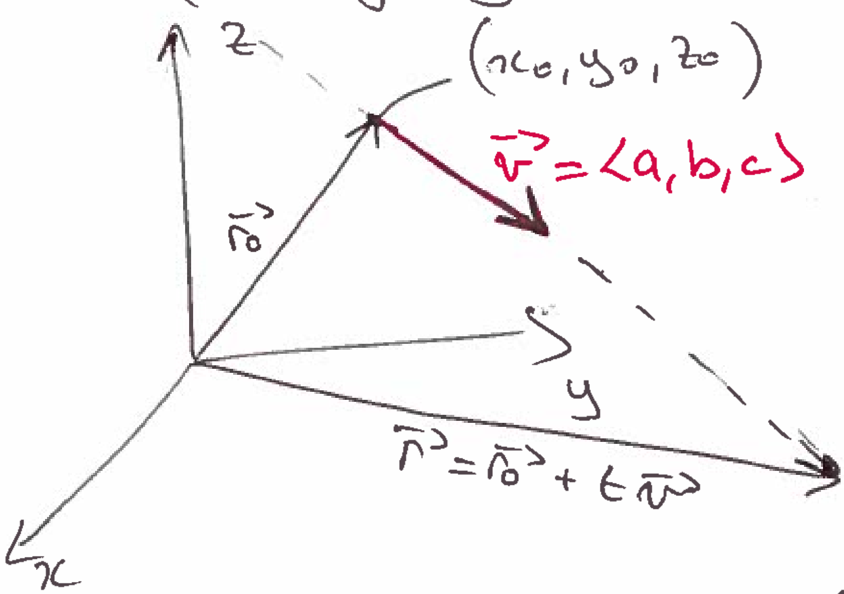
$$\cos\theta = \frac{\vec{n_1}\cdot\vec{n_2}}{|\vec{n_1}||\vec{n_2}|}$$

$|\vec{n_1}| = |\vec{n_2}| = \sqrt{3}$ and $\vec{n_1}\cdot\vec{n_2} = -1+1+1 = 1$ so

$$\cos\theta = \frac{1}{\sqrt{3}\sqrt{3}} = \frac{1}{3} \Rightarrow \theta = \cos^{-1}\left(\frac{1}{3}\right) \approx 70°$$

equations of lines and planes

a line in 3D is determined by a direction and a point it is going through.



t here is a parameter. Any vector $\vec{r}$ from the origin is of the form: $\boxed{\vec{r} = \vec{r_0} + t\vec{v}}$ for some t

So if (x,y,z) is a point on the line, there is some t such that

$$\langle x,y,z\rangle = \langle x_0, y_0, z_0\rangle + t\langle a,b,c\rangle$$
$$= \langle x_0+ta,\ y_0+tb,\ z_0+tc\rangle$$

a,b,c are the components of $\vec{v}$ ($\vec{v} = \langle a,b,c\rangle$).

<u>Parametric equation of the line:</u>

$$\boxed{\begin{aligned} x &= x_0 + ta \\ y &= y_0 + tb \\ z &= z_0 + tc \end{aligned}}$$

This is not unique, we could have chosen other (x_0, y_0, z_0) or any other vector $\vec{w}$ parallel to $\vec{v}$.

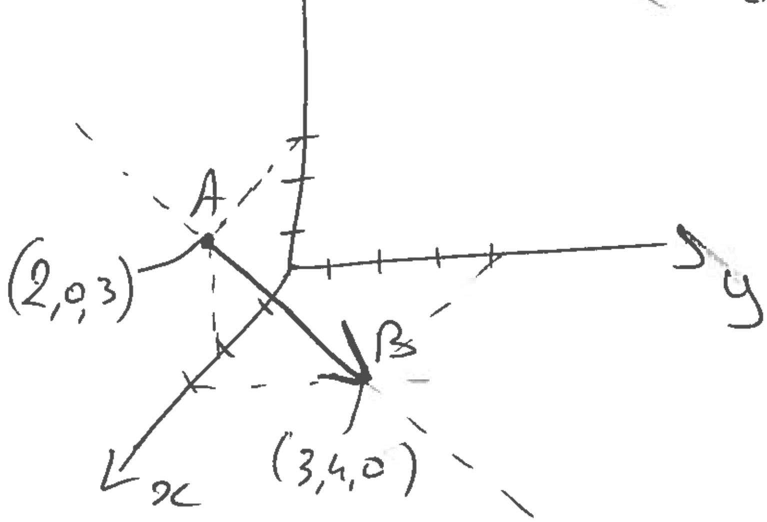
$\Rightarrow$ We can also eliminate t from the above equations to get 2 equations involving only x,y,z as variables (no t)

$$\boxed{\frac{x-x_0}{a} = \frac{y-y_0}{b} = \frac{z-z_0}{c}}$$

Sometimes called set of symmetric equations.

example : find the equations of the line passing through the points $(2,0,3)$ and $(3,4,0)$. (4)

let $(x_0, y_0, z_0) = (2,0,3)$



$\vec{v} = \vec{AB} = (3-2)\vec{i} + 4\vec{j} - 3\vec{k} \Rightarrow a = 1$

$b = 4$

$c = -3$

$$\Rightarrow \frac{x-2}{1} = \frac{y-0}{4} = \frac{z-3}{-3}$$

The first equation yields $\boxed{y = 4x - 8}$

The second " " $z = -\frac{3y}{4} + 3 = -\frac{3}{4}(4x-8) + 3$

$$\boxed{z = -3x + 9}$$

our line is the intersection of these two planes.

where does this line intersects the ⑤
y-z plane ?

$\rightsquigarrow$ y-z plane is defined by $x=0$

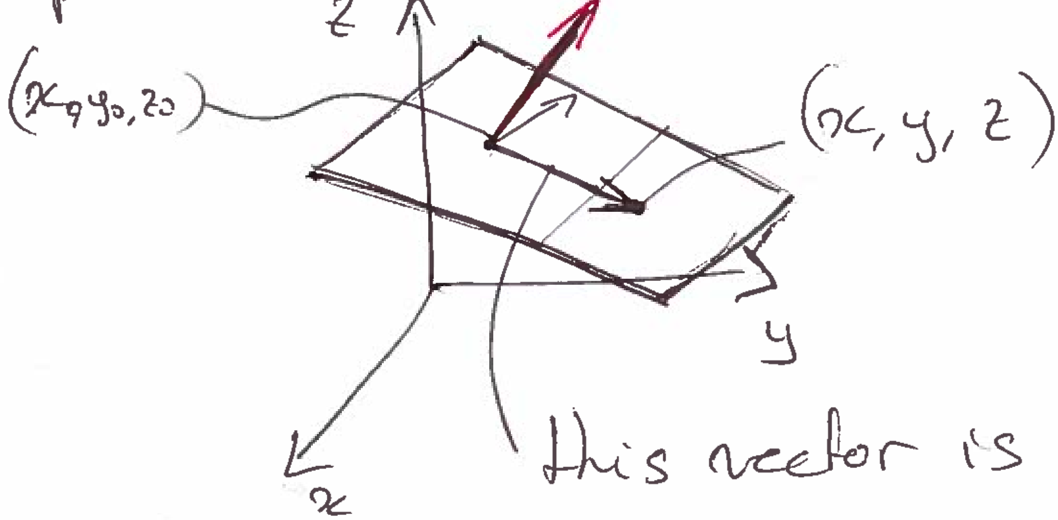
$\rightsquigarrow$ $\boxed{\begin{aligned} y &= -8 \\ z &= +9 \end{aligned}}$

The point where the line intersects the y-z plane is $(0,-8,9)$.

$\Rightarrow$ is there a similar way to determine a plane (with vectors) ?

$\Rightarrow$ A Plane is determined by a point on it (x_0, y_0, z_0) and a normal to this point

$\vec{n} = \langle a, b, c \rangle$



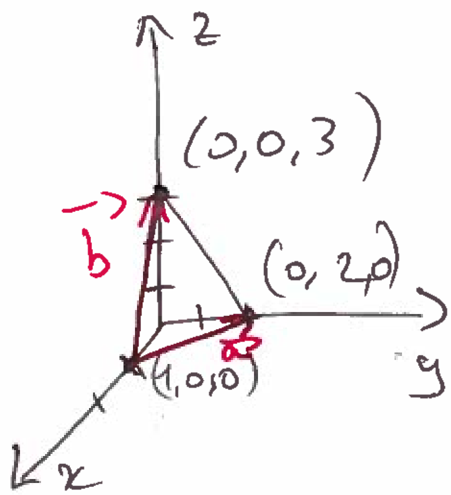
this vector is $\langle x-x_0, y-y_0, z-z_0 \rangle$

(x,y,z) is on the plane determined by $\vec{n}$ and (x_0, y_0, z_0) if and only if the vector $\langle x-x_0, y-y_0, z-z_0\rangle \cdot \langle a,b,c\rangle = 0$

$\Leftrightarrow$ $\boxed{ax + yb + cz + d = 0}$ $\vec{n} = \langle a,b,c\rangle$

with $d = -ax_0 - by_0 - cz_0$

Example: find the equation of the plane passing through the ~~planes~~ points $(0,0,3)$, $(0,2,0)$ and $(1,0,0)$



$\vec{a} = (-1,2,0)$

$\vec{b} = (-1,0,3)$

Compute $\vec{n} = \vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -1 & 2 & 0 \\ -1 & 0 & 3 \end{vmatrix} = 6\vec{i} + 3\vec{j} + 2\vec{k}$

any point (x, y, z) on the plane satisfies

$$6(x - x_0) + 3(y - y_0) + 2(z - z_0) = 0$$

pick $x_0 = 0$, $y_0 = 0$, $z_0 = 3$ or $x_0 = 0$, $y_0 = 2$, $z_0 = 0$ or $x_0 = 1$, $y_0 = 0$, $z_0 = 0$

$$\Rightarrow \boxed{6x - 6 + 3y + 2z = 0}$$