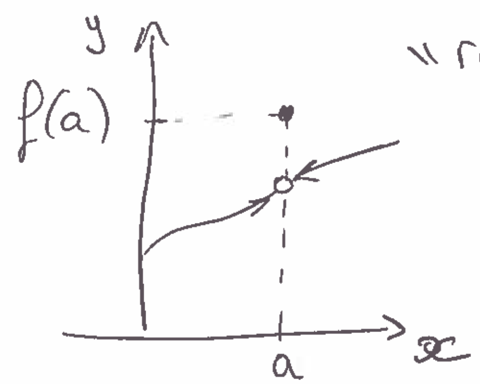
Continuity, differentiability

Calculus requires taking limits, which don't always exist. In a careful treatment of calculus, you define limits rigorously (using ε-δ) and you study criteria for exactly when limits exist. We will sweep most of this under the rug and discuss limits intuitively.

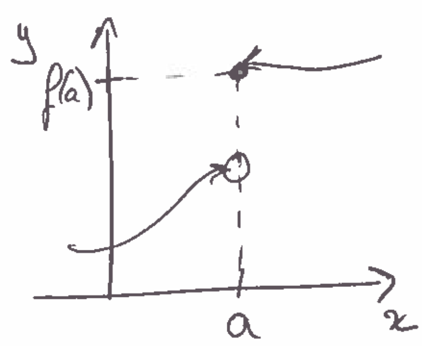
example of functions discontinuous:



« removal discontinuity »

limit exists but function is not continuous.

ie $\lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x)$ but $\lim_{x\to a} f(x) \neq f(a)$



« jump discontinuity »

limit does not exists

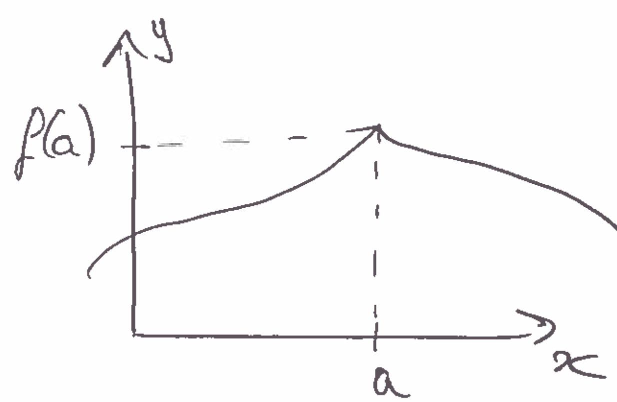
$\lim_{x\to a^-} f(x) \neq \lim_{x\to a^+} f(x)$

Continuity imposes:

$$\lim_{x\to a} f(x) = f(a)$$

you can show that differentiabily implies continuity (at a point) but the reverse statement is not always true.

example of continuous function at a point "a" but not differentiable (at "a")



$y=f(x)$ is continuous but not differentiable at "a" => $f'(a)$ does not exist.

=> Limits of two-variables functions

For a function of two variables, there is a new phenomenon: when looking at a limit

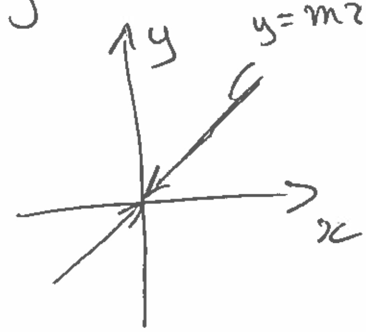
$$\lim_{(x,y)\to(0,0)} f(x,y)$$

there are many directions from which we can approach $(0,0)$. For the limit to exist, they must all agree.

Consider the example: $f(x,y)=\dfrac{xy}{x^2+y^2}$

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2}$$

Consider first the limit along the line $y = mx$



$$\Rightarrow \frac{xy}{x^2+y^2} = \frac{xmx}{x^2+(mx)^2} = \frac{x^2(m)}{x^2(1+m^2)} = \frac{m}{1+m^2}$$

$$\text{so } \lim_{(x,mx)\to(0,0)} f(x,y) = \lim_{(x,mx)\to(0,0)} \frac{m}{1+m^2} = \frac{m}{1+m^2}$$

the value of the limit is dependent of m, because of that the limit does not exist:

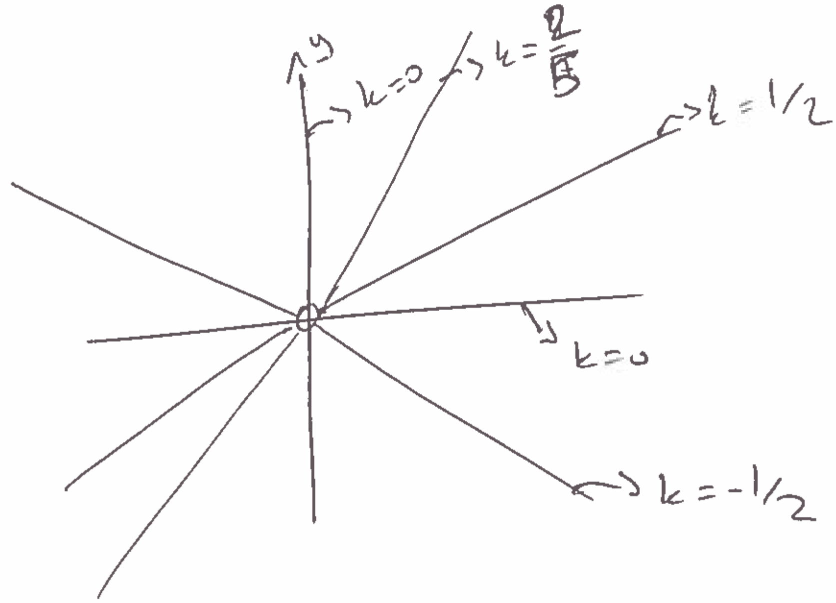
$$\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2} \text{ does not exist}$$

Actually, the origin is special here, at $(a,b) \neq (0,0)$ we do have:

$$\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2} = \frac{ab}{a^2+b^2}$$

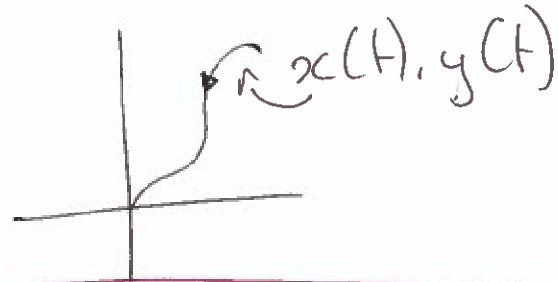
Contour lines disagree at origin

illustration



surface $z = \dfrac{xy}{x^2+y^2}$ has a vertical "crease" or z-axis.

we say $\lim\limits_{(x,y)\to(0,0)} f(x,y)$ exists if for all continuous paths $x = x(t)$ $y = y(t)$ with $x(0) = y(0) = 0$
$\lim\limits_{t\to 0} f(x(t), y(t))$ exists.



Definition: $f(x,y)$ is continuous at (x_0, y_0) if

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y) = f(x_0,y_0)$$

example $f(x,y) = \dfrac{xy}{x^2+y^2}$ $\begin{cases} \text{is not continuous at } (x,y) = (0,0) \\ \text{is continuous for all } (a,b) \neq (0,0) \end{cases}$

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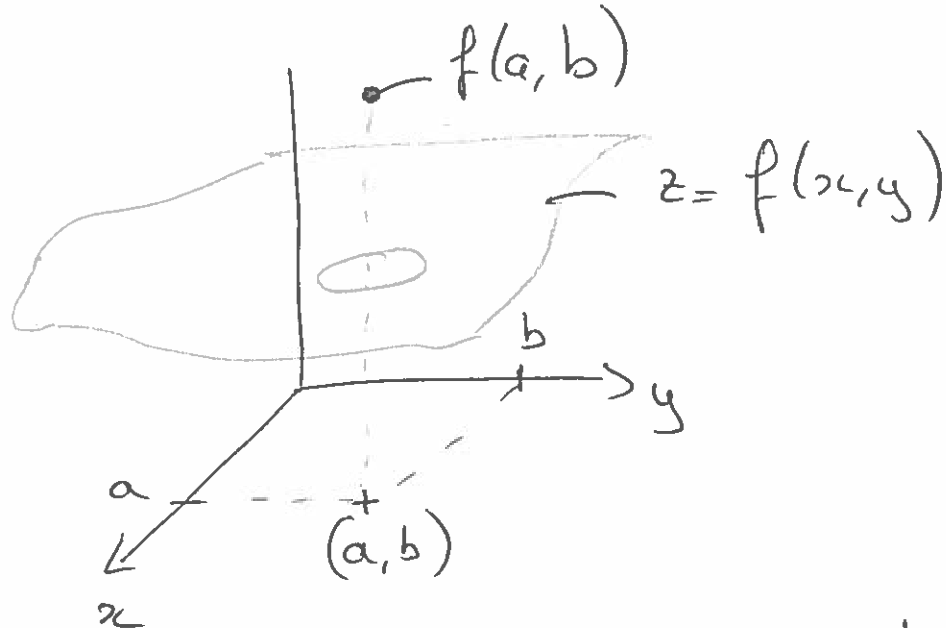
example



$\Rightarrow$ this function is not continuous at (a,b)