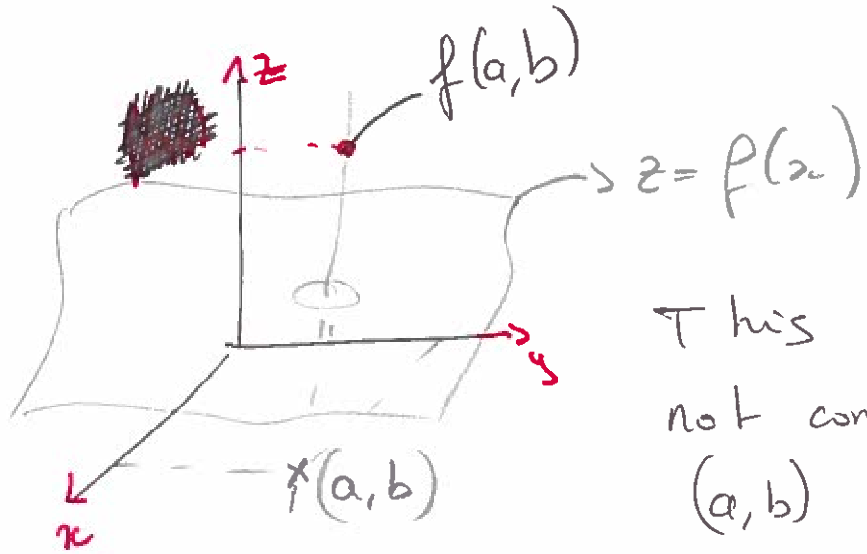
example: $f(x,y) = \dfrac{xy}{x^2+y^2}$ $\begin{cases} \text{is not continuous at } (0,0) \\ \text{is continuous for all } (a,b) \neq (0,0) \end{cases}$

example



This function is not continuous at (a,b)

Partial derivatives :

LECTURE 9

For a function $z = f(x,y)$ of 2 variables, if we fix one of variable, then we get a function of a single variable (the one not fixed), and we can take its derivative:

* for y fixed, we get $\dfrac{\partial z}{\partial x} = \dfrac{\partial f}{\partial x} = f_x \leftarrow$ function of (x,y)
* for x fixed, we get $\dfrac{\partial z}{\partial y} = \dfrac{\partial f}{\partial y} = f_y \leftarrow$ function of (x,y)

$\dfrac{\partial f}{\partial x}(x,y) = f_x(x,y)$ = derivative with respect to x, treating y as a constant

<u>Example</u> $f(x,y) = x^4 + 2x^2y^2 + y + e^{xy}$

$$f_x = \frac{\partial f}{\partial x} = 4x^3 + 4xy^2 + ye^{xy}$$

$$f_y = \frac{\partial f}{\partial y} = 4x^2y + 1 + xe^{xy}$$

$$f_x(1,2) = \frac{\partial f}{\partial x}(1,2) = 4(1)^3 + 4(1)(2)^2 + 2e^{1.2}$$
$$= 20 + e^2$$

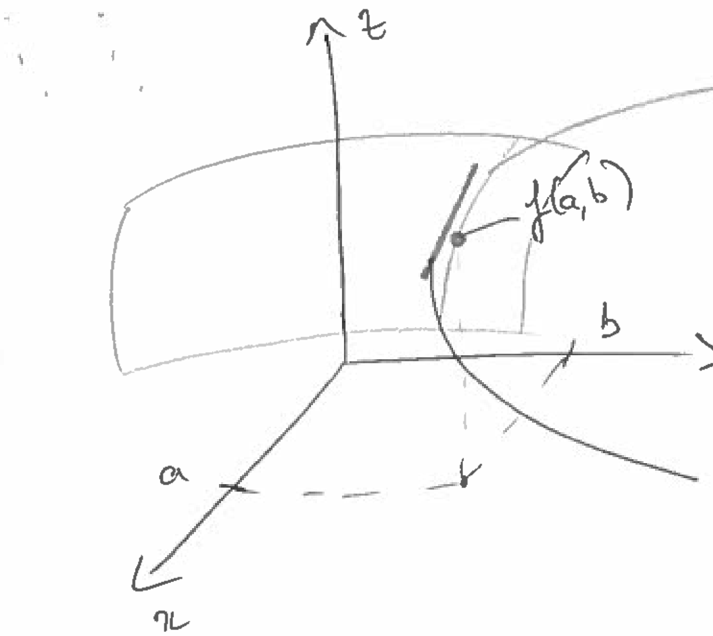
$$f_y(1,2) = \frac{\partial f}{\partial y}(1,2) = 4 \times 2 + 1 + e^2 = 9 + e^2$$

$\Rightarrow$ in terms of limits,

$$f_x(x,y) = \lim_{h \to 0} \frac{f(x+h,y) - f(x,y)}{h}$$

$$f_y(x,y) = \lim_{h \to 0} \frac{f(x,y+h) - f(x,y)}{h}$$

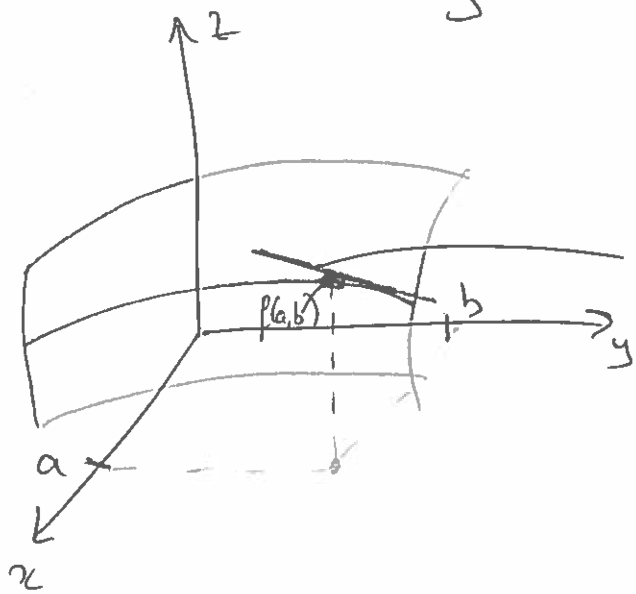
Geometric interpretation: slope of the trace curves



trace curve: intersection with plane $y = b$ (i.e y is fixed)

slope of tangent $= \dfrac{\partial f}{\partial x}(a,b)$

similar picture for $\dfrac{\partial f}{\partial y}(a,b)$



slope of tangent to trace curve is $\dfrac{\partial f}{\partial y}(a,b)$.

trace curve: intersection with plane $x = a$ (x is fixed)

Just as with one variable we have

implicit differentiation (4)

$\Rightarrow$ so if $z = \sqrt{1-x^2-y^2} \Rightarrow \dfrac{\partial z}{\partial x} = \dfrac{-x}{\sqrt{1-x^2-y^2}} = \dfrac{-x}{z}$

or if z is implicitly a function of x and y:

$$1 = x^2+y^2+z^2 \qquad \big(z = z(x,y)\big)$$

implicitly.

$$\frac{\partial(1)}{\partial x} = \frac{\partial}{\partial x}\left(x^2+y^2+z^2\right)$$

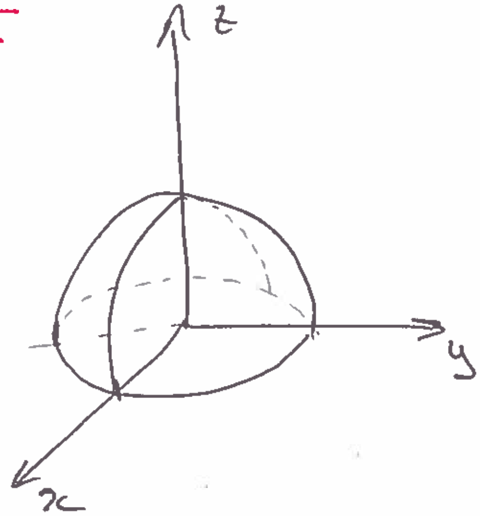
$$0 = 2x + 2z\frac{\partial z}{\partial x} \Rightarrow \text{so } \frac{\partial z}{\partial x} = \frac{-x}{z} \quad \text{(as before)}$$

Compare geometry

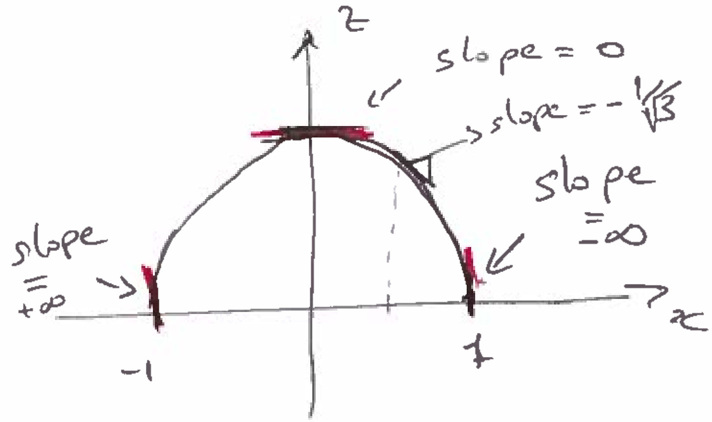
$$x^2+y^2+z^2=1$$

$$\Rightarrow z=\sqrt{1-x^2-y^2}$$

$$\frac{\partial z}{\partial x}=\frac{-x}{\sqrt{1-x^2-y^2}}$$



look for example at trace curve corresponding to $y=0$



$$\frac{\partial z}{\partial x}(1,0)=\frac{-1}{\sqrt{1-(1^2)-0^2}}=\frac{-1}{0}=-\infty$$

$$\frac{\partial z}{\partial x}(0,0)=\frac{-0}{\sqrt{1-(0)^2-(0)^2}}=0$$

$$\frac{\partial z}{\partial x}(-1,0)=\frac{-(-1)}{\sqrt{1-(-1)^2-(0)^2}}=\frac{1}{0}=+\infty$$

$$\frac{\partial z}{\partial x}(1/2,0)=\frac{-(1/2)}{\sqrt{1-(1/2)^2-(0)^2}}=\frac{-1/2}{\sqrt{3/4}}=\frac{-1}{2\sqrt{3/(2^2)}}=\frac{-1}{\frac{2}{2}\sqrt{3}}=\frac{-1}{\sqrt{3}}$$

Example

$$f(x,y) = \int_y^x g(t)\,dt$$

(Fundamental theorem of calculus) $\Rightarrow \dfrac{\partial f}{\partial x} = g(x)$ and $\dfrac{\partial f}{\partial y} = -g(y)$

Higher derivatives

$$f_{xx} = \frac{\partial}{\partial x}\left(\frac{\partial}{\partial x} f(x,y)\right) = \frac{\partial^2}{\partial x} f(x,y)$$

$$f_{xy} = \frac{\partial}{\partial x}\left(\frac{\partial}{\partial y} f\right) = \frac{\partial^2}{\partial x \partial y} f(x,y)$$

$$f_{yy} = \frac{\partial}{\partial y}\left(\frac{\partial}{\partial y} f\right) = \frac{\partial^2}{\partial y^2} f(x,y)$$

$$f_{yx} = \frac{\partial}{\partial y}\left(\frac{\partial}{\partial x} f\right) = \frac{\partial^2}{\partial y \partial x} f(x,y)$$

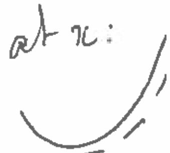
Theorem: $f_{xy} = f_{yx}$

order doesn't matter.

Recall from single-variable calculus:

$f \uparrow \Rightarrow f' > 0$

$f''(x) > 0$: $f' \uparrow$ at x :



Concave up

$f' \searrow$

$f'' < 0$ concave downwards

$\Rightarrow$ bigger f'' means greater curvature.