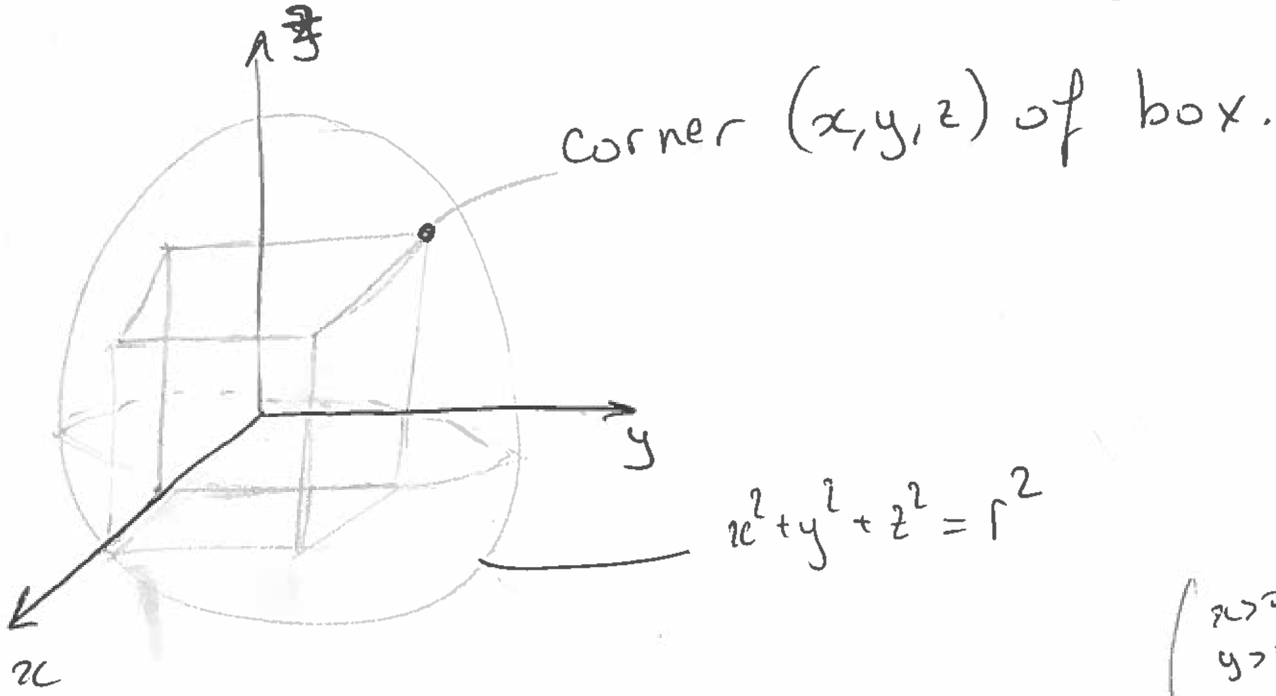
Example: Find the maximum volume of a rectangular box that is constrained to be inscribed in a sphere of radius r.

$\boxed{\text{Lecture 20}}$



$$\left(\begin{array}{l} x>0 \\ y>0 \\ y>0 \end{array}\right)$$

Volume of the box is $V = 2x\,2y\,2z$

constrat: $x^2+y^2+z^2=r^2$

so $z=\sqrt{r^2-x^2-y^2}$

then $V = 8xy\sqrt{r^2-x^2-y^2}$.

$$\Rightarrow \frac{\partial V}{\partial x} = 8y\sqrt{r^2-x^2-y^2} + 8xy\,\frac{1}{2}\frac{(-2x)}{\sqrt{r^2-x^2-y^2}}$$

$$\frac{\partial V}{\partial x} = \frac{8y(r^2-x^2-y^2)-8x^2y}{\sqrt{r^2-x^2-y^2}} = \frac{8y(r^2-2x^2-y^2)}{\sqrt{r^2-x^2-y^2}}$$

$$\frac{\partial V}{\partial y} = 8x\sqrt{r^2-x^2-y^2} + \frac{8xy(-2y)}{2\sqrt{r^2-x^2-y^2}}$$

$$\frac{\partial V}{\partial y} = \frac{8x(r^2-x^2-y^2) - 8y^2x}{\sqrt{r^2-x^2-y^2}} = \frac{8x(r^2-x^2-2y^2)}{\sqrt{r^2-x^2-y^2}}$$

$\Rightarrow$ critical points: $\frac{\partial V}{\partial x} = \frac{\partial V}{\partial y} = 0$

$\Rightarrow \frac{\partial V}{\partial x} = 0$ gives $y=0$ or $\boxed{r^2 - 2x^2 - y^2 = 0}$ ①

$\frac{\partial V}{\partial y} = 0$ gives $x=0$ or $\boxed{r^2 - x^2 - 2y^2 = 0}$ ②

↳ cannot give max volume.

↳ $2x^2+y^2 = 2y^2+x^2 \Rightarrow y^2 = x^2 \Rightarrow x=y$

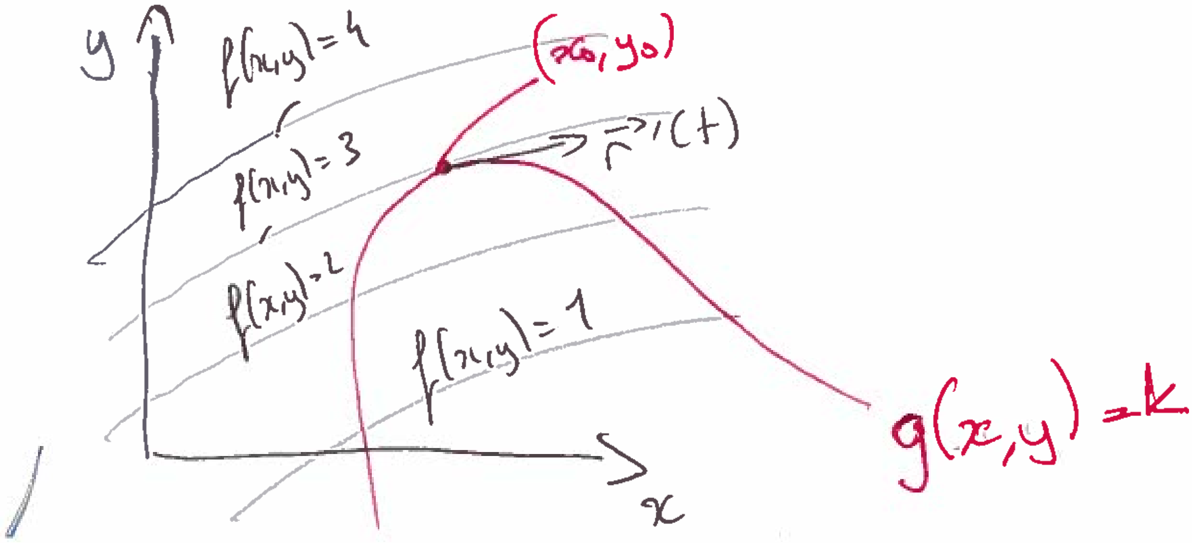
$\Rightarrow r^2 = 2x^2 + x^2 \Rightarrow r^2 = 3x^2 \Rightarrow x = \frac{r}{\sqrt{3}} = y$

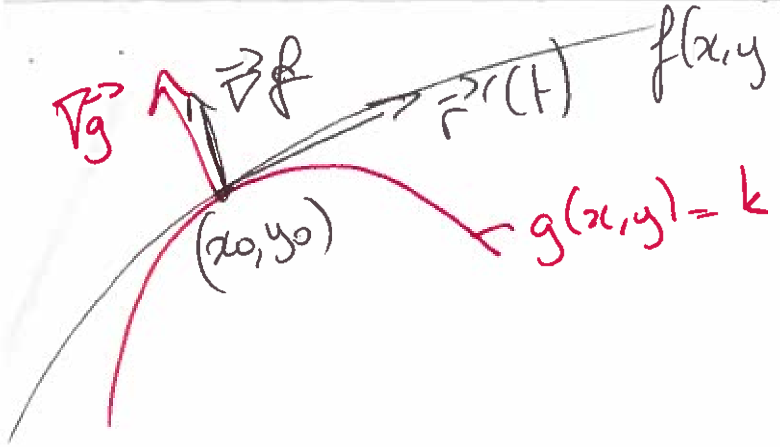
(use ① or ②)

only one critical point (with $x,y>0$) : $(x,y) = \left(\frac{r}{\sqrt{3}}, \frac{r}{\sqrt{3}}\right)$

max volume is $V = \frac{8r^2}{3}\sqrt{r^2 - \frac{r^2}{3} - \frac{r^2}{3}} = \frac{8r^3}{3\sqrt{3}}$

⇒ Methode of Lagrange multipliers.





More precisely, let $\vec{r}(t) = (x(t), y(t))$ describe the contour $g(x,y) = k$. Then if $f(\vec{r}(t))$ is extrema we have

$$0 = \frac{d}{dt} f(x(t), y(t)) = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} = \vec{\nabla} f \cdot \vec{r}\,' \quad \text{at } t = t_0$$

$\Rightarrow$ $\vec{\nabla} f(x_0, y_0)$ is orthogonal to the contour line $g(x,y) = k$. But $\vec{\nabla} g(x_0, y_0)$ is also orthogonal to $g(x,y) = k$. So we must have $\vec{\nabla} f = \lambda \vec{\nabla} g$

This works also for more variables:

To find Extreme values of $f(x,y,z)$ given $g(x,y,z) =$

$\Rightarrow$ solve $\begin{cases} \vec{\nabla} f = \lambda \vec{\nabla} g \\ g(x,y,z) = k \end{cases}$ $\begin{cases} \text{4 equations with} \\ \text{4 unknowns.} \end{cases}$

Finding extreme values of $f(x,y)$ subject to a constraint $g(x,y)=k$.

$\Rightarrow$ From picture, extreme value occurs at (x_0,y_0) where contour of f is tangent to contour $g(x,y)=k$.

$\Rightarrow$ i.e where $\vec{\nabla} f$ is parallel to $\vec{\nabla} g$

$\Rightarrow$ i.e where $\vec{\nabla} f(x_0,y_0) = \lambda \vec{\nabla} g(x_0,y_0)$

$\Rightarrow$ This gives 3 eqns for 3 unknowns : (x,y,λ)

namely:

$$\frac{\partial f}{\partial x}(x,y) = \lambda \frac{\partial g}{\partial x}(x,y)$$

$$\frac{\partial f}{\partial y}(x,y) = \lambda \frac{\partial g}{\partial y}(x,y)$$

$$g(x,y) = k$$

Example (Again) Maximize $V = 8xyz = f(x,y,z)$

subject to $x^2+y^2+z^2-r^2=0 = g(x,y,z)$

$\Rightarrow \vec{\nabla} f = \langle 8yz, 8xz, 8xy \rangle$, $\vec{\nabla} g = \langle 2x, 2y, 2z \rangle$

$$\Rightarrow \text{solve} \begin{cases} 8yz = \lambda 2x \\ 8xz = \lambda 2y \\ 8xy = \lambda 2z \\ x^2+y^2+z^2 = r^2 \end{cases}$$

$$\Rightarrow \frac{y}{x} = \frac{x}{y} \Rightarrow x^2 = y^2 \Rightarrow x = y$$

$$\Rightarrow \frac{\cancel{8}z}{\cancel{8}y} = \frac{y}{z} \Rightarrow z^2 = y^2 \Rightarrow z = y$$

$$\Rightarrow \boxed{x = y = z}$$

$$3x^2 = r^2 \Rightarrow x = \frac{r}{\sqrt{3}} \qquad 8\frac{r}{\sqrt{3}}\frac{\cancel{r}}{\cancel{\sqrt{3}}} = 2\lambda \frac{\cancel{r}}{\cancel{\sqrt{3}}}$$

$$\boxed{\lambda = \frac{4r}{\sqrt{3}}}$$

and $V_{max} = 8\frac{r}{\sqrt{3}}\frac{r}{\sqrt{3}}\frac{r}{\sqrt{3}} = \frac{8r^3}{3\sqrt{3}}$

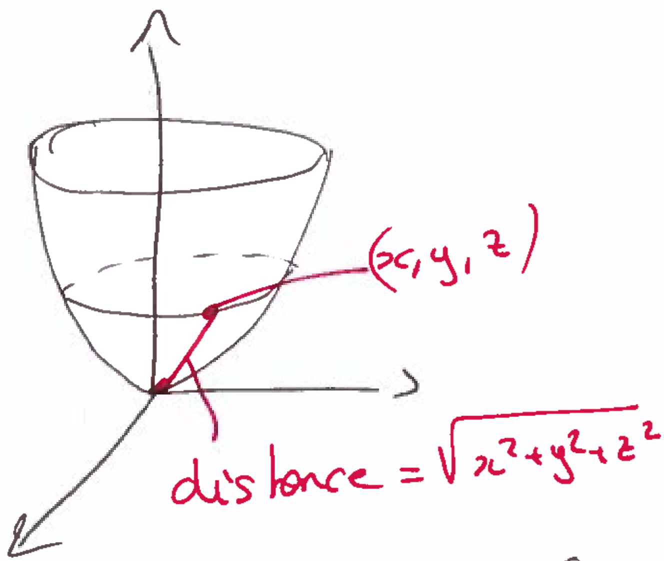
as before.

Example The plane $x+y+2z=2$ intersects the paraboloid $z=x^2+y^2$ in an ellipse. Find the points on the ellipse nearest and farthest from the origin.

max/min for

$$\text{distance}^2 = x^2+y^2+z^2$$

with $z^2=(x^2+y^2)^2$



$$\Rightarrow f(x,y) = x^2+y^2+(x^2+y^2)^2$$

constraint is $g(x,y) = x+y+2\underbrace{(x^2+y^2)}_{=z}-2=0$

$\Rightarrow$ Solve

$$f_x = \lambda g_x$$
$$f_y = \lambda g_y$$
$$g = 0$$

$$f_x = 2x+2(x^2+y^2)2x$$

$$f_y = 2y+2(x^2+y^2)2y$$

$$g_x = 1+4x$$

$$g_y = 1+4y$$

$$2x + 4x(x^2+y^2) = \lambda(1+4x) \quad (\times y)$$ LHS equals.

$$2y + 4y(x^2+y^2) = \lambda(1+4y) \quad (\times x)$$

$$\Rightarrow 2xy + 4xy(x^2+y^2) = \lambda y(1+4x)$$

$$\Rightarrow 2xy + 4xy(x^2+y^2) = \lambda x(1+4x$$

$$\Rightarrow \boxed{y = x}$$

constraint g becomes:

$$2x + 4x^2 - 2 = 0 \quad \Rightarrow 2x^2 + x - 1 = 0$$

$$\Rightarrow (2x-1)(x+1) = 0$$

$$x = \frac{1}{2} \text{ or } x = -1$$

$\Rightarrow$ extreme points are $(1/2, 1/2)$ and $(-1,-1)$

↳ corresponding z values are $\left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2 = \frac{1}{2}$ and $(-1)^2 + (-1)^2 = 2$

$$\Rightarrow \text{Distance } \sqrt{x^2+y^2+z^2} = \begin{cases} \sqrt{(1/2)^2 + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2} = \frac{\sqrt{3}}{4} & \Rightarrow (1/2, 1/2, 1/2) \text{ closest} \\ \sqrt{1+1+4} = \sqrt{6} & \Rightarrow (-1,-1,+2) \text{ farthest} \end{cases}$$

⚠ have a look Problem 11 Webwork 6