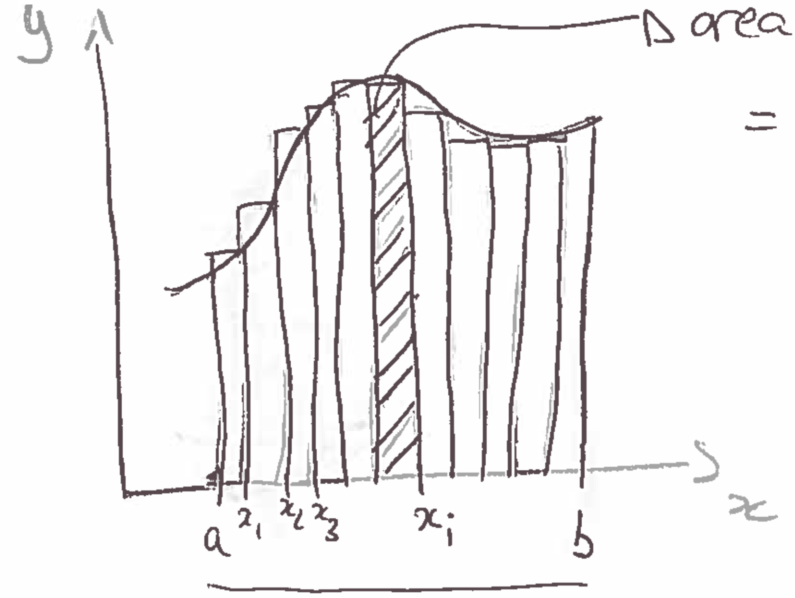
Doubles Integrals over Rectangles domains

Recall basics for single veriable integrals.



area of rectangle is $= f(x_i)\Delta x$

partition of the domain $[a,b]$

$\Delta x = x_i - x_{i-1}$ (for any i)

$\Delta x = \dfrac{b-a}{N}$ ← the number of regtangles.

$\Rightarrow$ Riemann sum : $\displaystyle\sum_{i=1}^{N} f(x_i)\Delta x$ sum of areas of rectangle

$\Rightarrow$ Definition of integrals

$$\int_a^b f(x)\,dx = \lim_{N\to\infty} \sum_{i=1}^{N} f(x_i)\Delta x$$

limit exists if $f(x)$ is continious and is the same value even if we replace $f(x_i)$ by $f(x_i^*)$ fo any $x_i^* \in [x_{i-1}, x_i]$.

i.e: we could have taken x_i to be any poin in the ith interval (left, right, midpoint, whateve in the limit it doesn't matter.

$\Rightarrow$ Area is an interpretation of the integral (not the goal of integration). Another interpretati is in terms of average value of a function.

$$\frac{1}{N}\sum_{i=1}^{N} f(x_i) \approx \text{average} \qquad \left(N = \frac{b-a}{\Delta x}\right)$$

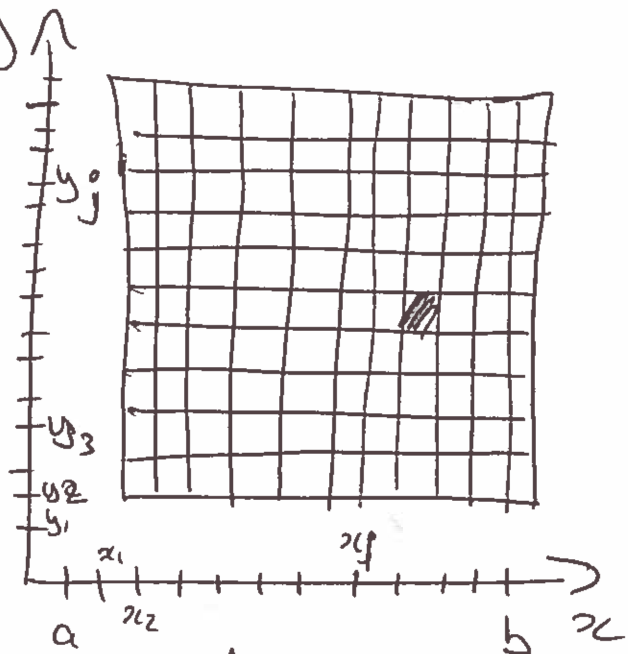
$$= \frac{1}{b-a}\sum_{i=1}^{N} f(x_i)\,\Delta x.$$

So $\displaystyle\frac{1}{b-a}\int_a^b f(x)\,dx \equiv$ average value of $f(x)$

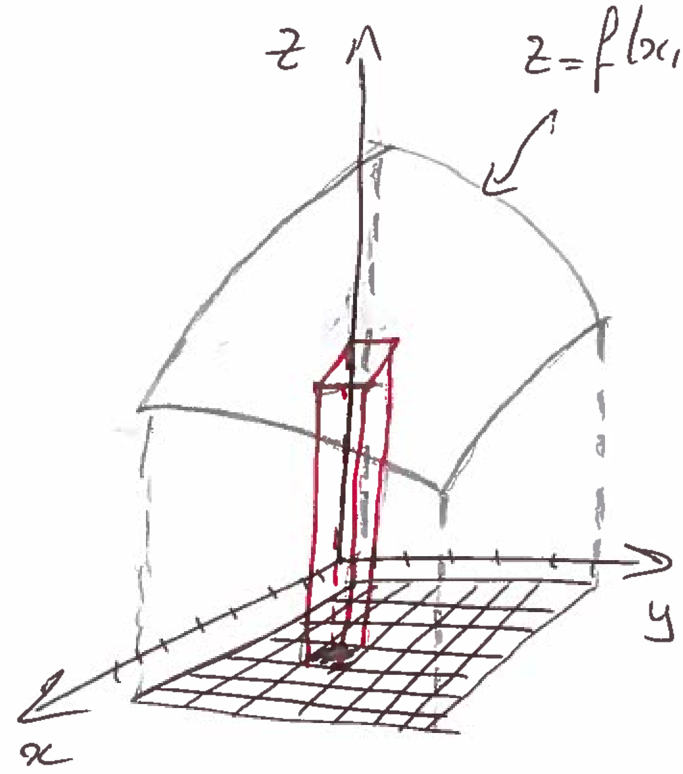
over $[a, b]$.

Two variables $f(x,y)$ $(x,y) \in D = [a,b] \times [c,d]$

$\Delta y = \frac{d-c}{M}$

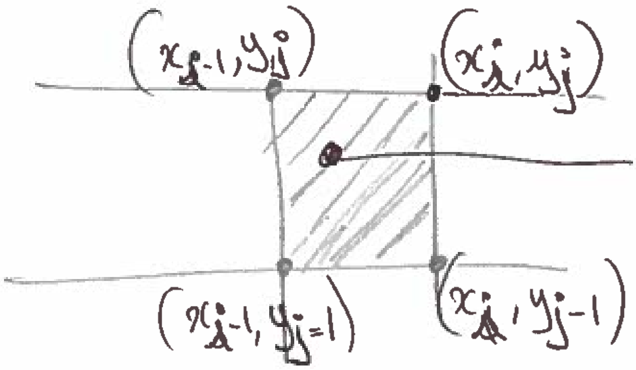


$\Delta x = \frac{b-a}{N}$



Volume under graph $\approx \sum_{i=1}^{N} \sum_{j=1}^{M} f(x_i, y_j) \Delta x \Delta y$

$$\Rightarrow \iint_R f(x,y)\,dx\,dy = \lim_{\substack{M \to \infty \\ N \to \infty}} \sum_{i=1}^{N} \sum_{j=1}^{M} \underbrace{f(x_i, y_j)}_{\text{or } f(x_i^*, y_j^*)} \Delta x \Delta y$$



(x_i^*, y_j^*) ← any point in the (i,j)th rectangle. Could be midpoint, corner....

Average value

$$\text{Average value over } R \approx \frac{1}{M}\,\frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{M} f(x_i^*, y_j^*)$$

$$= \underbrace{\overset{M}{\frac{1}{(d-c)}}\;\overset{N}{\frac{1}{(b-a)}}}_{\frac{1}{\text{area}(R)}}\sum_{i=1}^{N}\sum_{j=1}^{M} f(x_i^*, y_j^*)\,\Delta x\,\Delta y$$

$\Rightarrow$ So we define $f_{average} = \dfrac{1}{\text{area}(R)} \displaystyle\iint_R f(x,y)\,dx\,dy$.

$\Rightarrow$ **Properties**:

linearity.

$$\iint_R [f(x,y) + g(x,y)]\,dx\,dy = \iint_R f(x,y)\,dx\,dy + \iint_R g(x,y)\,dx\,dy$$

$$\iint_R c\,f(x,y)\,dx\,dy = c\iint_R f(x,y)\,dx\,dy \quad (c \text{ is a constant})$$

monotonicity: if $f(x,y) \leq g(x,y)$

then $\displaystyle\iint_R f(x,y)\,dx\,dy \leq \iint_R g(x,y)\,dx\,dy$.