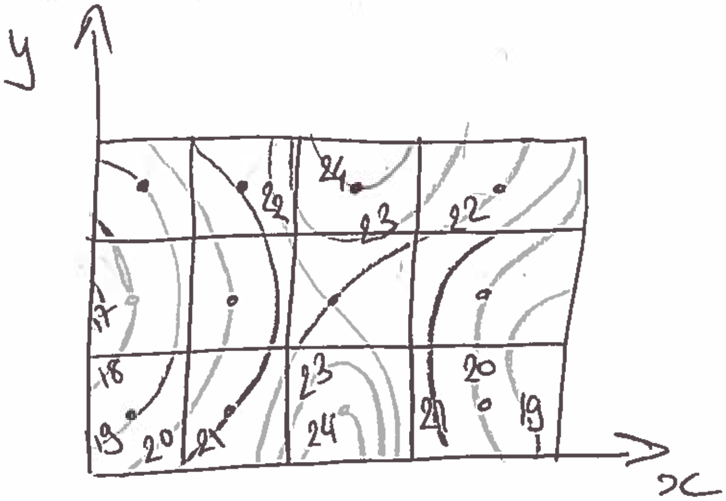
Example: Approximate the average temp in a region whose contour plot is: $\boxed{\text{Lecture 22}}$ (5)



use midpoint and a 4x3 grid.

$$\Rightarrow \text{Average temp} \approx \frac{1}{12}\left[19+18+19+21+20+21 + 24+22+24+20+20+22\right]$$

$$= 20{,}833\ldots$$

Iterated integrals.

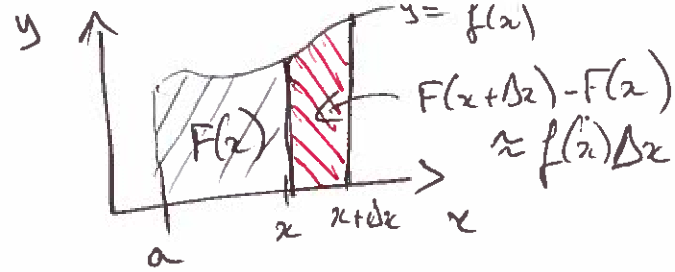
$\Rightarrow$

For single-variable calculus, we compute $\int_a^b f(x)dx$ using Fundamental theorem of calculus:

if $F' = f$ then $\int_a^b f(x)dx = F(b) - F(a)$

$\Rightarrow$ Reduces the problem of integration to a problem of finding antiderivatives (which is nice! because there are many techniques for this).

Idea of proof:



6

Let $F(x) = \int_a^x f(t)\,dt$

then $\dfrac{F(x+\Delta x) - F(x)}{\Delta x} = \dfrac{1}{\Delta x} \underbrace{\int_x^{x+\Delta x} f(t)\,dt}_{\approx f(x)\Delta x} \to f(x)$ as $\Delta x \to 0$

So $F'(x) = f(x)$.

$\Rightarrow$ For 2 variables: $f(x,y)$ $\quad (x,y) \in [a,b] \times [c,d]$

Given x, consider $A(x) = \underbrace{\int_c^d f(x,y)\,dy}$ ← partial integral.

integral treats x as a constant, answer depends on x, (limits are for y).

$\Rightarrow$ Iterated integral:

$$\int_a^b A(x)\,dx = \int_a^b \left[\int_c^d f(x,y)\,dy \right] dx$$

write as $\displaystyle\int_a^b\int_c^a f(x,y)\,dy\,dx$

$\Rightarrow$ the other iterated integral:

$$\int_c^d\int_a^b f(x,y)\,dx\,dy = \int_c^d\left[\int_a^b f(x,y)\,dx\right]dy$$

$\Rightarrow$ <u>Fubini theorem</u> : it works in $\left\{\begin{array}{l}\text{either}\\ \text{both}\end{array}\right.$ ways:

$$\iint_R f(x,y)\,dx\,dy = \int_a^b\int_c^d f(x,y)\,dy\,dx = \int_c^d\int_a^b f(x,y)\,dx\,dy$$

for $R = \{[a,b]\times[c,d]\}$

<u>example</u>:

$$\int_{y=0}^{y=1}\left[\int_{x=0}^{x=1} x\left(y+x^2\right)^2 dx\right]dy = \int_{y=0}^{y=1}\left[\frac{1}{2\cdot 3}\left(y+x^2\right)^3\right]_0^1 dy$$

$$= \frac{1}{6}\int_{y=0}^{y=1}\left((y+1)^3 - y^3\right)dy = \frac{1}{6}\left[\frac{1}{4}(y+1)^4 - \frac{y^4}{4}\right]_0^1$$

$$= \frac{1}{24}\left[\left(2^4-1\right)-\left(1^4-0\right)\right] = \frac{14}{24} = \frac{7}{12}$$

$\Rightarrow$ the other way:

$$\int_{x=0}^{x=1}\left[\int_{y=0}^{y=1} x\left(y+x^2\right)^2 dy\right] dx = \int_{x=0}^{x=1}\left[\frac{x}{3}\left(y+x^2\right)^3\right]_0^1 dx$$

$$= \frac{1}{3}\int_{x=0}^{x=1} x\left(1+x^2\right)^3 - x^7 \, dx = \frac{1}{3}\left[\frac{\left(1+x^2\right)^4}{4 \cdot 2} - \frac{x^8}{8}\right]_0^1$$

$$= \frac{1}{3}\left\{\frac{2^4}{8} - \frac{1}{8} - \left(\frac{1^4}{8} - 0\right)\right\} = \frac{14}{24} = \frac{7}{12}$$