for $(x_0, y_0) = (0,0)$ $\quad z_0 = e^0 = 1$ $\quad$ Lecture 12

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$$\frac{\partial z}{\partial x} = -2xe^{-x^2-y^2} \Rightarrow \frac{\partial z}{\partial x}(0,0) = 0$$

$$\frac{\partial z}{\partial y}(0,0) = 0$$

$z = z_0 = 1$ is the equation of the tangent plane at $(x_0, y_0) = (0,0)$

Def The "Linear approximation" or "tangent plane approximation" to f at (a,b) is

$$f(x,y) \approx f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$$

example: Find the linear approximation to $f(x,y)$ and use it to approximate

$f(x,y) = \sqrt{x^2+y^2}$ at $(3,4)$

$\sqrt{(3.1)^2 + (3.9)^2}$

$$f_x = \frac{x}{\sqrt{x^2+y^2}} \; ; \; f_y = \frac{y}{\sqrt{x^2+y^2}}$$

$$\Rightarrow f(3,4) = \sqrt{3^2+4^2} = 5$$

$$f_x(3,4) = \frac{3}{5} \; ; \; f_y(3,4) = \frac{4}{5}$$

$$\Rightarrow f(3.1, 3.9) \approx 5 + \frac{3}{5}(3.1-3) + \frac{4}{5}(3.9-4)$$

$$= 5 + \frac{3}{50} - \frac{4}{50} = 4.98 00$$

exact value is 4.98192

notation $\Delta x = x - a$, $\Delta y = y - b$, $\Delta z = f(x,y) - f(a,b)$

$= f(\Delta x + a, \Delta y + b) - f(a$

The linear approximation becomes:

$$\boxed{\Delta z \approx f_x(a,b)\,\Delta x + f_y(a,b)\,\Delta y}$$

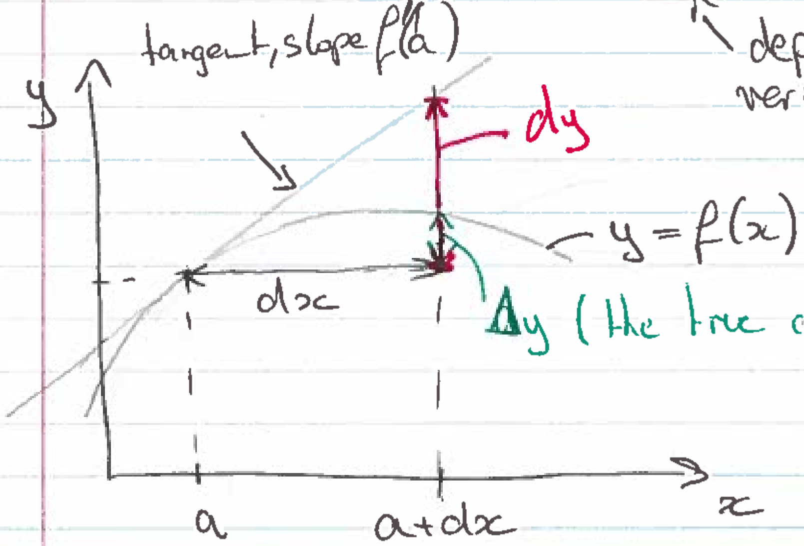
Approximation gets better as $\Delta x, \Delta y \to 0$
(because then $\Delta z \to 0$)

<u>Differentials</u>: In single variable calculus, for

$y = f(x)$ we have $dy = f'(x)\,dx$

dependent variable

independent variable.



so $f(a+dx) = f(a) + \Delta y$ but Δy is unknown, so we
use $\Delta y \approx dy$ with $dy = f'(a)\,dx$

In two variables case, for $z = f(x,y)$ we define :

independent variable

dependent variable — $dz = f_x(x,y)dx + f_y(x,y)dy$

$$dz = \frac{\partial f}{\partial x}(x,y)dx + \frac{\partial f}{\partial y}(x,y)dy$$

or similarly $\boxed{df = \frac{\partial f}{\partial x}(x,y)dx + \frac{\partial f}{\partial y}(x,y)dy}$

Example: A rectangular box has dimensions $80 \times 100 \times 50$ cm, each measured to within 1 mm.
=> Estimate the maximum error in te volume.

Solution: Volume $V = xyz$

$$dV = \frac{\partial V}{\partial x}dx + \frac{\partial V}{\partial y}dy + \frac{\partial V}{\partial z}dz = yz\,dx + xz\,dy + xy\,dz$$

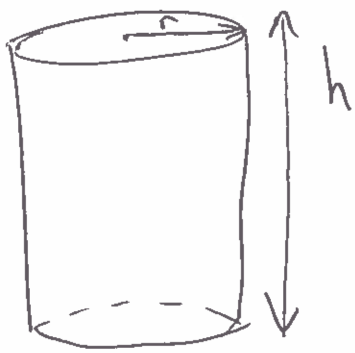
$$dV = (100 \times 50)(0{,}1) + (80 \times 50)(0{,}1) + (80 \times 100)(0{,}1)\ cm^3$$

$$= 500 + 400 + 800 = 1700\ cm^3$$

=> Max error is $\pm 1700\ cm^3$ => relative max error $\frac{1700}{80 \times 100 \times 50} = 0{,}4\ \%$

<u>Example</u> (chapter 14.4 n°34 Apex)

Estimate the amount of metal in a closed cylindrical can, 10cm high, 4 cm in diameter, if top and bottom are 0,1cm thick and sides 0,05cm thick.



Volume $V = \pi r^2 h$

$$dV = \frac{\partial V}{\partial r} dr + \frac{\partial V}{\partial h} dh$$

$$dV = \underbrace{2r\pi h}_{\text{surface area of the sides}} dr + \underbrace{\pi r^2}_{\text{surface area of top (or bottom)}} dh$$

$$dV = 2(2)\pi(10)(0{,}05) + \pi(2)^2(0{,}1+0{,}1)$$

$$dV = \frac{40\pi}{20} + 4\pi(0{,}2) = 2\pi + 0{,}8\pi = 2{,}8\pi \approx 8{,}8\,cm^3$$

<u>Material for midterm 1 ends here</u> ⚠