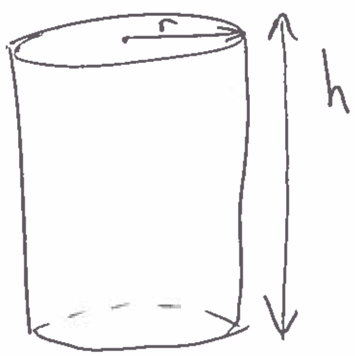
Example (chapter 14.4 n°34 Apex) $\boxed{\text{Lecture 14}}$

Estimate the amount of metal in a closed cylindrical can, 10cm high, 4 cm in diameter, if top and bottom are 0,1cm thick and sides 0,05cm thick.



Volume $V = \pi r^2 h$

$$dV = \frac{\partial V}{\partial r} dr + \frac{\partial V}{\partial h} dh$$

$$dV = \underbrace{2r\pi h}_{\text{surface area of the sides}} dr + \underbrace{\pi r^2}_{\text{surface area of top (or bottom)}} dh$$

$$dV = 2(2)\pi(10)(0{,}05) + \pi(2)^2(0{,}1+0{,}1)$$

$$dV = \frac{40\pi}{20} + 4\pi(0{,}2) = 2\pi + 0{,}8\pi = 2{,}8\pi \approx 8{,}8\,cm^3$$

Material for midterm 1 ends here ⚠

Exact calculation for can example:

$$V(r+dr, h+dh) - V(r,h) = \pi(r+dr)^2(h+dh) - \pi r^2 h$$

$$= \pi\left\{(r^2+2rdr+dr^2)(h+dh) - r^2h\right\}$$

$$= \pi\left\{\cancel{r^2h} + r^2(dh) + 2rh(dr) + 2r(dr)(dh) + (dr)^2(dh) + h(dr)^2 - \cancel{r^2h}\right\}$$

$$= \underbrace{\pi r^2(dh) + 2\pi rh(dr)}_{\substack{\text{Linear terms} \\ \parallel \\ dV}} + \overbrace{\underbrace{2r(dr)(dh) + h(dr)^2}_{\substack{\text{2nd order} \\ \text{Quadratic terms}}} + \underbrace{(dr)^2(dh)}_{\substack{\text{3rd order} \\ \text{terms.}}}}$$

These are the error in the linear approximation (as they are missing). Higher order terms in dr, dh. Corrects for what is missing in linear approximation

Chain rule $y = f(g(t))$ $\frac{dy}{dt} = f'(g(t))\, g'(t)$

here $y = f(x)$ with $x = g(t)$

$$\boxed{\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt}}$$

$\Rightarrow$ This is the single variable chain rule.

$\Rightarrow$ more variables:

$z = f(g(t), h(t))$ again $z = f(x,y)$ with $x = g(t)$ and $y = h(t)$.

$$\boxed{\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}}$$

recall differential

$$dz = \frac{\partial z}{\partial x} dz + \frac{\partial z}{\partial y} dy$$

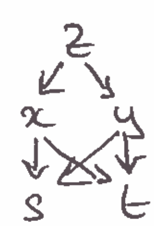
if $z = f(x,y)$ with $x = g(s,t)$ and $y = h(s,t)$

then $\frac{\partial z}{\partial s}$ and $\frac{\partial z}{\partial t}$ are given by

$$\boxed{\begin{aligned} \frac{\partial z}{\partial s} &= \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s} \\ \frac{\partial z}{\partial t} &= \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t} \end{aligned}}$$

The derivatives with respect to s are now partial because we "fixed" t.

dependency graph



example : switching to polar coordinates.

$z = f(x,y) \quad x = r\cos\theta \quad y = r\sin\theta$

$$\frac{\partial z}{\partial r} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial r}$$

$$\frac{\partial z}{\partial \theta} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial \theta}$$

$$\frac{\partial x}{\partial r} = \cos\theta \qquad \frac{\partial y}{\partial r} = \sin\theta \qquad \frac{\partial x}{\partial \theta} = -r\sin\theta \qquad \frac{\partial y}{\partial \theta} = r\cos\theta$$

$$\frac{\partial z}{\partial r} = \frac{\partial z}{\partial x}\cos\theta + \frac{\partial z}{\partial y}\sin\theta$$

$$\frac{\partial z}{\partial \theta} = -\frac{\partial z}{\partial x} r\sin\theta + \frac{\partial z}{\partial y} r\cos\theta$$

problem : compute $\left(\frac{\partial z}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial z}{\partial \theta}\right)^2$ in terms of $\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}$

$$\Rightarrow \left(\frac{\partial z}{\partial x}\cos\theta + \frac{\partial z}{\partial y}\sin\theta\right)^2 + \frac{1}{r^2}\left(-\frac{\partial z}{\partial x} r\sin\theta + \frac{\partial z}{\partial y} r\cos\theta\right)^2$$

$$= \left(\frac{\partial z}{\partial x}\right)^2\left(\cos^2\theta + \sin^2\theta\right) + \left(\frac{\partial z}{\partial y}\right)^2\left(\sin^2\theta + \cos^2\theta\right)$$

$$+ 2\frac{\partial z}{\partial x}\frac{\partial z}{\partial y}\cos\theta\sin\theta - \frac{2}{r^2} r\sin\theta\, r\cos\theta \frac{\partial z}{\partial x}\frac{\partial z}{\partial y}$$

$$\Rightarrow \left(\frac{\partial z}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial z}{\partial \theta}\right)^2 = \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2$$

Chain rule and implicit differentiation

We can use the chain rule to better understand implicit differentiation. Suppose we have an equation $F(x,y,z)=0$, if we can explicitly solve this equation for z, we get $z=f(x,y)$, some function of (x,y). Maybe we can't solve it explicitly but we can regard $F(x,y,z)=0$ as $z=f(x,y)$ such that

$$F(x,y,f(x,y))=0.$$

example $2x+3y-4z-e^{xyz-1}=0$

$\Rightarrow$ can't solve for z. We can get $\frac{\partial z}{\partial x}=f_x$ and $f_y=\frac{\partial z}{\partial y}$ by differentiations $F(x,y,z)=0$ with respect to x or y.