$$\Rightarrow \left(\frac{\partial z}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial z}{\partial \theta}\right)^2 = \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2$$

Chain rule and implicit differentiation

Lecture 15

We can use the chain rule to better understand implicit differentiation. Suppose we have an equation $F(x,y,z)=0$, if we can explicitly solve this equation for z, we get $z=f(x,y)$, some function of (x,y).

Maybe we can't solve it explicitly but we can regard $F(x,y,z)=0$ as defining $z=f(x,y)$ such that

$F(x,y,f(x,y))=0$.

example $2x+3y-4z-e^{xyz-1}=0$

$\Rightarrow$ can't solve for z. We can get $\frac{\partial z}{\partial x}=f_x$ and $f_y=\frac{\partial z}{\partial y}$ by differentiating $F(x,y,z)=0$ with respect to x or y.

$$\frac{\partial(F(x,y,z))}{\partial x} = \frac{\partial(0)}{\partial x} = 0$$

indep variables   dependent variables.

because y is independent of x

$$\frac{\partial(F(x,y,z))}{\partial x} = \frac{\partial F}{\partial x}\underbrace{\frac{\partial x}{\partial x}}_{=1} + \frac{\partial F}{\partial y}\cancel{\frac{\partial y}{\partial x}}^{\,0} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial x} = 0$$

it comes that:

$$\frac{\partial F}{\partial x} = -\frac{\partial F}{\partial z}\frac{\partial z}{\partial x} \quad \text{or} \quad \boxed{\frac{\partial z}{\partial x} = -\frac{\partial F}{\partial x} \Big/ \frac{\partial F}{\partial z}}$$

similarly

$$\boxed{\frac{\partial z}{\partial y} = -\frac{\partial F}{\partial y} \Big/ \frac{\partial F}{\partial z}}$$

In the example $F(x,y,z) = 2x + 3y - 4z(x,y) - e^{xyz-} =$

$$\frac{\partial(F(x,y,z))}{\partial x} = \frac{\partial}{\partial x}\left(2x + 3y + -4z(x,y) - e^{xy\,z(x,y)-1}\right) = 0$$

$$\frac{\partial F(x,y,z)}{\partial x} = 2 - 4\frac{\partial z}{\partial x} - \left(yz\,e^{xyz-1} + xy\,e^{xyz-1}\frac{\partial z}{\partial x} \right) = 0$$

it comes that:

$$\boxed{\frac{\partial z}{\partial x} = \frac{yz\,e^{xyz-1} - 2}{-4 - xy\,e^{xyz-1}}}$$

similarly:

$$\frac{\partial z}{\partial y}\left(2x + 3y - 4z - e^{xyz-1}\right) = 0$$

it comes

$$3 - 4\frac{\partial z}{\partial y} - e^{xyz-1}\left(xz + xy\frac{\partial z}{\partial y}\right) = 0$$

$$\frac{\partial z}{\partial y}\left(-4 - xy\,e^{xyz-1}\right) = e^{xyz-1}\cdot xz - 3$$

$$\boxed{\frac{\partial z}{\partial y} = \frac{xz\,e^{xyz-1} - 3}{-4 - xy\,e^{xyz-1}}}$$

Find now the equation of the plane tangent to the surface $2x+3y-4z-e^{xyz-1}=0$ at the point $(x,y,z)=(1,1,1)$

$\Rightarrow$ check first that $(1,1,1)$ is on the surface

$$2+3-4-1=0 \quad \underline{\underline{ok}}$$

$\Rightarrow$ the equation of the plane is

$$z_p = z_0 + \frac{\partial z}{\partial x}(x_0,y_0)(x-x_0) + \frac{\partial z}{\partial y}(x_0,y_0)(y-y_0)$$

$$\Rightarrow z_p = 1 + \left(\frac{1-2}{-4-1}\right)(x-1) + \left(\frac{1-3}{-4-1}\right)(y-1)$$

$$\Rightarrow \boxed{z_p = 1 + \frac{1}{5}(x-1) + \frac{2}{5}(y-1)}$$

Find an approximation solution to the equation

$$\frac{5}{3}+\frac{7}{2}-4z-e^{\frac{35}{36}z-1}=0$$

$\Rightarrow$ hint: consider $F(x,y,z)$ for $(x,y) = \left(\frac{5}{6}, \frac{7}{6}\right)$

since (x,y) is close to $(1,1)$, the tangent plane is a good approximation

$$z_p\left(\frac{5}{6}, \frac{7}{6}\right) = 1 + \frac{1}{5}\left(\frac{5}{6} - 1\right) + \frac{2}{5}\left(\frac{7}{6} - 1\right)$$

$$= 1 + \frac{1}{5}\left(-\frac{1}{6}\right) + \frac{2}{5}\left(\frac{1}{6}\right) = 1 + \frac{1}{30} = \frac{31}{30}$$

$$\boxed{z\left(\frac{5}{6}, \frac{7}{6}\right) \approx z_p\left(\frac{5}{6}, \frac{7}{6}\right) = \frac{31}{30}}$$