Apex chapter 14.6 Lecture 16

Directional Derivatives and gradient vector

example: a hill has altitude $z = f(x,y) = e^{-x^2-y^2}$

and you drive with position $x = t$, $y = 1-t$.

Your height at time t is $f(x(t), y(t))$ and

your rate of ascent is

(* plot figure here)

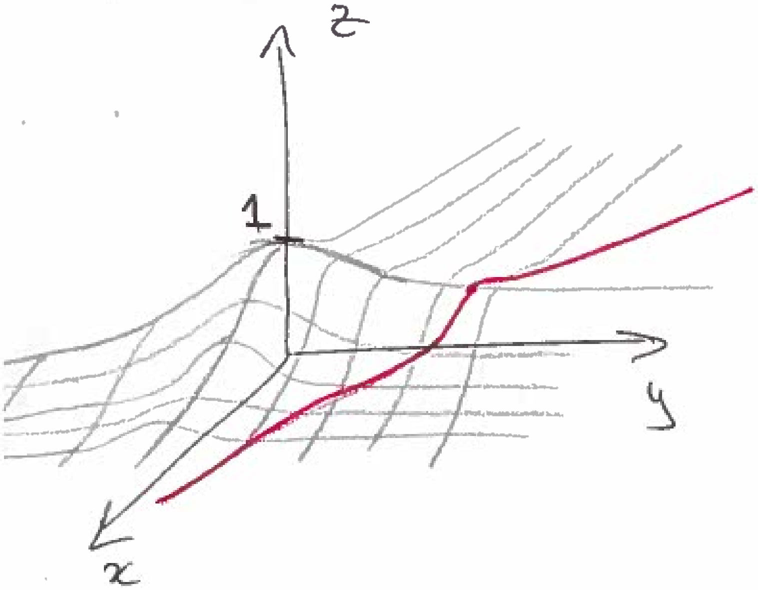
$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

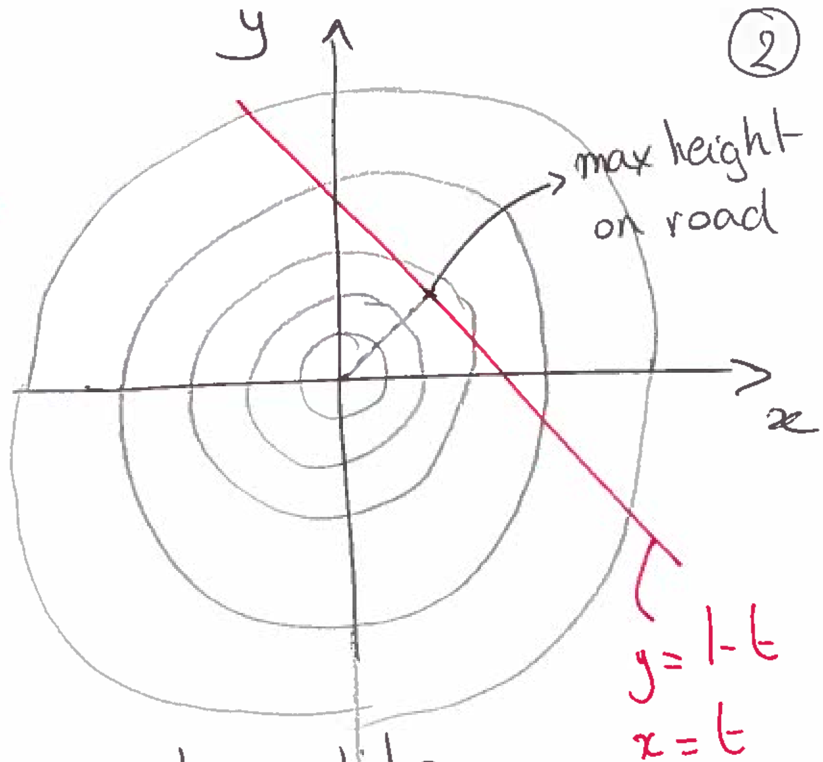
$$= -2xe^{-x^2-y^2}\frac{dx}{dt} + -2ye^{-x^2-y^2}\frac{dy}{dt}$$

$$= -2xe^{-x^2-y^2} + 2ye^{-x^2-y^2}$$

$$\frac{dz}{dt} = 2e^{-x^2-y^2}(y-x)$$

when is $z(t)$ maximum? (When are we at the heighest point of the road?





contour plots
of $z = f(x,y)$

$\Rightarrow$ maximum height occurs when $\frac{dz}{dt} = 0$, i.e $y = x$

which means $1 - t = t$ so $t = \frac{1}{2}$

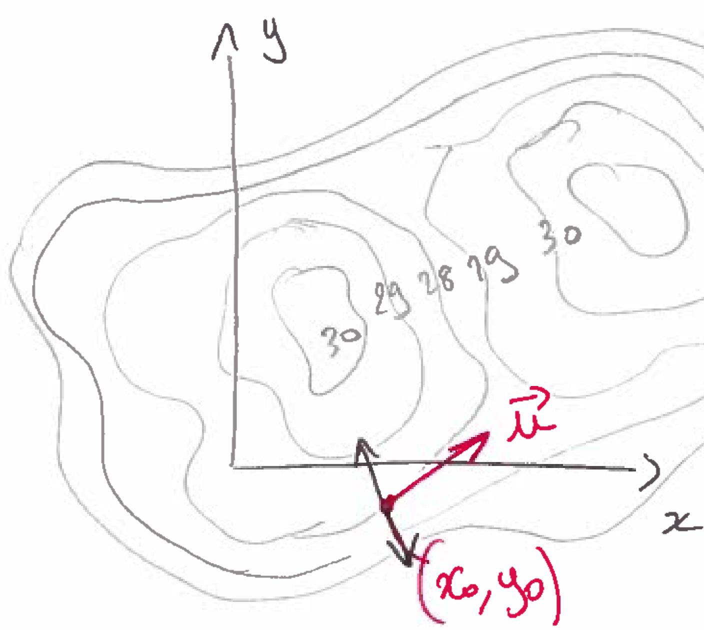
the maximum height location is then $(x_{max}, y_{max}) = (\frac{1}{2}, \frac{1}{2})$

and $z_{max} = f(x_{max}, y_{max}) = e^{-1/2} = \frac{1}{\sqrt{e}} \approx 0{,}60$

what we have done here is the idea of

the directional derivative

and we are at some location (x_0, y_0). It doesn't make sense to ask how fast does $f(x,y)$ change at (x_0, y_0)? But it does make sense to ask how fast $f(x,y)$ changes as we move in some direction $\vec{u}$, (with $|\vec{u}| = 1$).

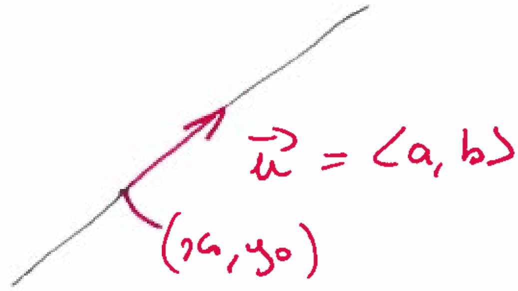


change of elevation depends on the direction of the road.

$(D_{\vec{u}} f)(x_0, y_0)$ = rate of change of $f(x,y)$ as we move past (x_0,y_0) in the direction of $\vec{u}$ at a constant speed of 1.

$\vec{u} = \langle a, b \rangle \qquad |\vec{u}| = 1.$

$$\left.\begin{aligned} x &= x_0 + at \\ y &= y_0 + bt \end{aligned}\right\}$$ parametric equations for the line through (x_0, y_0) in the direction of $\vec{u}$.



then $$(D_{\vec{u}} f)(x_0, y_0) = \frac{d f}{dt}(0) = \left(\frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t}\right)\Bigg|_{t=0}$$

$$\frac{\partial f}{\partial x}(x_0,y_0)\frac{\partial x(0)}{\partial t} + \frac{\partial f}{\partial x}(x_0,y_0)\frac{\partial y(0)}{\partial t} = \frac{\partial f}{\partial x}(x_0,y_0)\, a + \frac{\partial f}{\partial y}(x_0,y_0)\, b$$

$$(D_{\vec{u}} f)(x_0,y_0) = \left\langle \frac{\partial f}{\partial x}(x_0,y_0), \frac{\partial f}{\partial y}(x_0,y_0) \right\rangle \cdot \underbrace{\langle a, b \rangle}_{\vec{u}}$$

$\Rightarrow$ We found that $D_{\vec{u}} f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle \cdot \vec{u}$

⚠ Definition: The gradient of a function f of x, y is $\overrightarrow{\text{grad}}\, f = \vec{\nabla} f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle$

Thus, we have:

$$\boxed{D_{\vec{u}} f = \vec{\nabla} f \cdot \vec{u}}$$

$\Rightarrow$ we compute the gradient of f and from that we can easily compute any directional derivative. Special cases $D_{\vec{i}} f = \frac{\partial f}{\partial x}$, $D_{\vec{j}} f = \frac{\partial f}{\partial y}$

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Example: Find the gradient of $f(x,y) = x^2 + y^2$
(infinite paraboloid)
and compute the directional derivative $(D_{\vec{u}} f)(x_0, y_0)$
$x_0 = \sqrt{2}$ and $y_0 = \sqrt{2}$
for $\forall \vec{u}_1 = \frac{1}{\sqrt{2}} \langle 1, 1 \rangle$, $\vec{u}_2 = \frac{1}{\sqrt{2}} \langle 1, -1 \rangle$, $\vec{u}_3 = \langle 1, 0 \rangle$

$\Rightarrow \frac{\partial f}{\partial x} = 2x \mid \frac{\partial f}{\partial y} = 2y$, so $\vec{\nabla} f(x_0, y_0) = \langle 2x_0, 2y_0 \rangle$

$\vec{\nabla} f(\sqrt{2}, \sqrt{2}) = \langle 2\sqrt{2}, 2\sqrt{2} \rangle$

$$(D_{\vec{u}_1} f)(\sqrt{2}, \sqrt{2}) = \langle 2\sqrt{2}, 2\sqrt{2} \rangle \cdot \frac{1}{\sqrt{2}} \langle 1, 1 \rangle$$

$$= 4$$

$\Rightarrow$ The function increases in the direction of $\vec{u}_1$ at a rate of 4.