Thus, we have:

$$\boxed{D_{\vec{u}} f = \vec{\nabla} f \cdot \vec{u}}$$

$\Rightarrow$ we compute the gradient of f and from that we can easily compute any directional derivative. Special cases $D_{\vec{i}} f = \frac{\partial f}{\partial x}$, $D_{\vec{j}} f = \frac{\partial f}{\partial y}$

Lecture 17

Example: Find the gradient of $f(x,y) = x^2 + y^2$ (infinite paraboloid) and compute the directional derivative $(D_{\vec{u}} f)(x_0, y_0)$

$x_0 = \sqrt{2}$ and $y_0 = \sqrt{2}$

for $\forall \vec{u}_1 = \frac{1}{\sqrt{2}} \langle 1, 1 \rangle$, $\vec{u}_2 = \frac{1}{\sqrt{2}} \langle 1, -1 \rangle$, $\vec{u}_3 = \langle 1, 0 \rangle$

$\Rightarrow \frac{\partial f}{\partial x} = 2x \;\; \Big| \;\; \frac{\partial f}{\partial y} = 2y$, so $\vec{\nabla} f(x_0, y_0) = \langle 2x_0, 2y_0 \rangle$

$\vec{\nabla} f(\sqrt{2}, \sqrt{2}) = \langle 2\sqrt{2}, 2\sqrt{2} \rangle$

$$(D_{\vec{u}_1} f)(\sqrt{2}, \sqrt{2}) = \langle 2\sqrt{2}, 2\sqrt{2} \rangle \cdot \frac{1}{\sqrt{2}} \langle 1, 1 \rangle$$

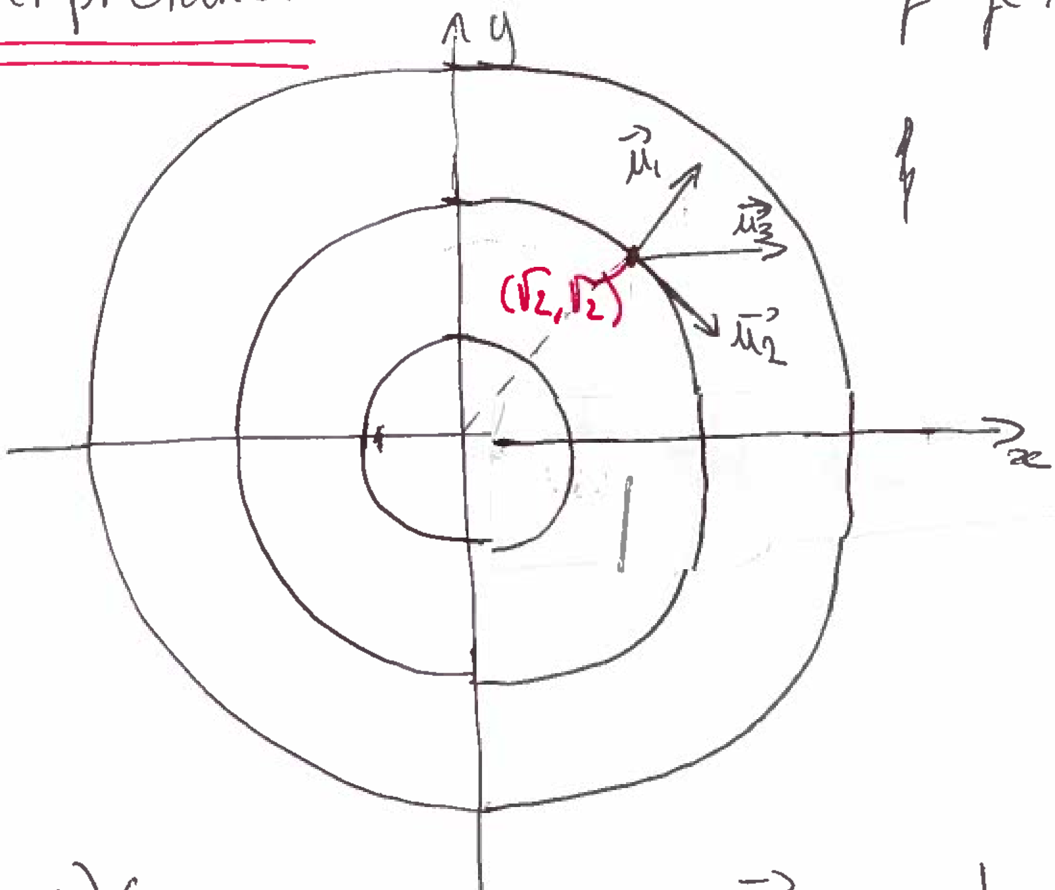
$$= 4$$

$\Rightarrow$ The function increases in the direction of $\vec{u}_1$ at a rate of 4.

$$\left(D_{\vec{u}_2} f\right)(\sqrt{2},\sqrt{2}) = \langle 2\sqrt{2}, 2\sqrt{2}\rangle \cdot \frac{1}{\sqrt{2}}\langle 1,-1\rangle = 0$$

$$\left(D_{\vec{u}_3} f\right)(\sqrt{2},\sqrt{2}) = \langle 2\sqrt{2}, 2\sqrt{2}\rangle \cdot \langle 1,0\rangle = 2\sqrt{2}$$

Interpretation: level curves of $f(x,y)$



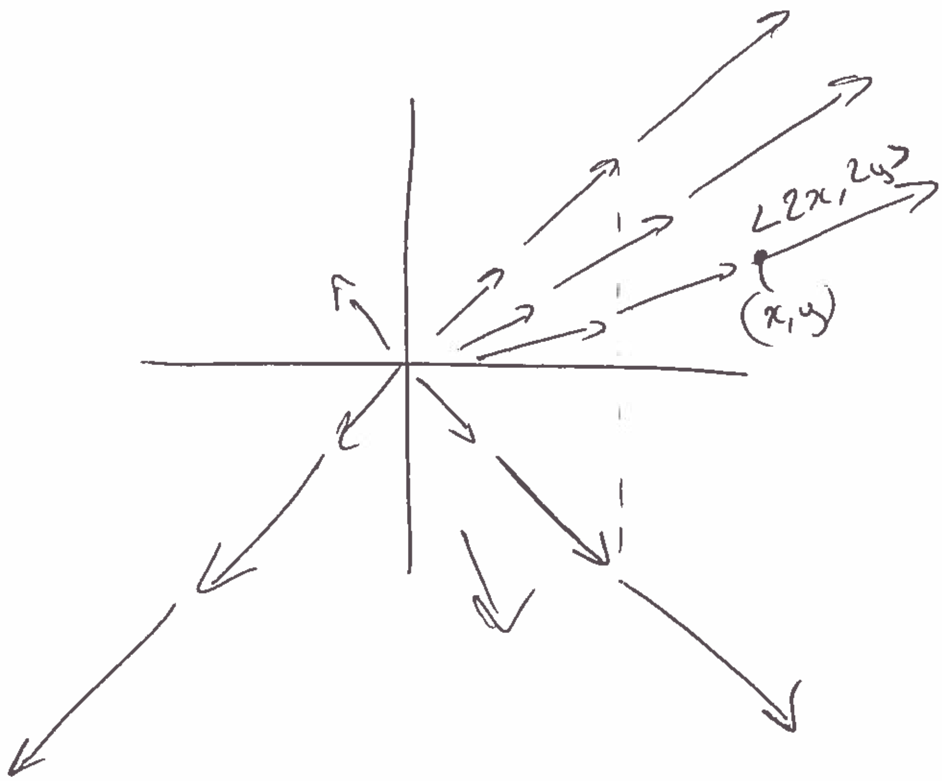
$\left(D_{\vec{u}_2} f\right)(\sqrt{2},\sqrt{2}) = 0$ because $\vec{u}_2$ is tangent to the contour line.

Also, note that $\vec{\nabla} f(\sqrt{2},\sqrt{2})$ is orthogonal to the contour line.

The gradient defines a vector field:

here $\vec{\nabla} f = \langle 2x, 2y \rangle$

at each point (x,y), draw the vector $\vec{\nabla} f$.

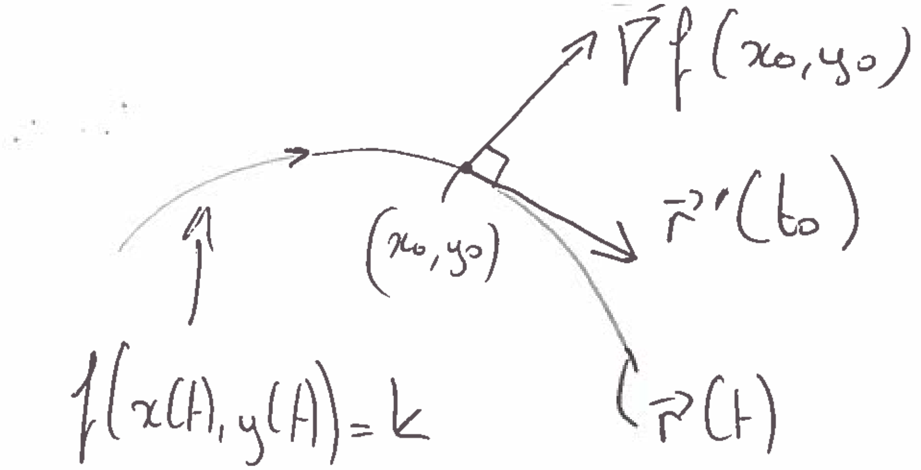


The gradient of f at (x_0, y_0) is always normal to the contour line of f through (x_0, y_0). To see this, suppose the contour has value k and let

$\vec{r}(t) = \langle x(t), y(t) \rangle$ be the contour, with

$\vec{r}(t_0) = \langle x_0, y_0 \rangle$. Then $\vec{r}\,'(t) = \langle x'(t), y'(t) \rangle$

is tangent to the contour at $\vec{r}(t)$



We obtain: $\dfrac{d}{dt} f(x(t), y(t)) = \dfrac{dk}{dt} = 0$

and by the chain rule.

$$\frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} = 0 \quad , \text{ ie } \boxed{\vec{\nabla} f \cdot \vec{r}\,'(t) = 0}$$

At $t = t_0$ $\quad \vec{\nabla} f(x_0, y_0) \cdot \vec{r}\,'(t_0) = 0$ so $\vec{\nabla} f \perp$ contour.

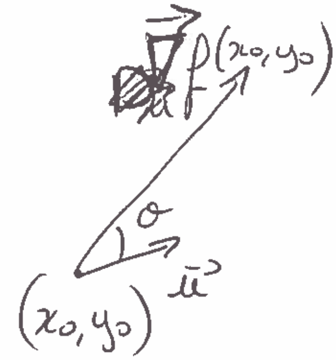
Maximum rate of change of $z = f(x, y)$:

For $\vec{u}$ with $|\vec{u}| = 1$ we have

$$(D_{\vec{u}} f)(x_0, y_0) = \vec{\nabla} f(x_0, y_0) \cdot \vec{u}$$

so $D_{\vec{u}} f(x_0, y_0) = |\vec{\nabla} f(x_0, y_0)| \, \underbrace{|\vec{u}|}_{=1} \cdot \cos\theta$

this is the largest when $\cos\theta = 1$ i.e $\theta = 0$, i.e $\vec{u}$ is in the same direction as $\vec{\nabla} f$.



so $\vec{\nabla} f$ points in the direction of maximum rate of change and $|\vec{\nabla} f|$ is that rate of change.

gradients points up the steepest part of the hill. And they are perpendicular to the contour lines.

Gradient and Directional derivatives of functions of 3-variables

Similar ideas apply for 3 or more variables.

For $w = F(x,y,z)$ we define $\vec{\nabla} F = \left(\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z} \right)$ and the directional derivative is $D_{\vec{u}} F = \vec{\nabla} F \cdot \vec{u}$ with $\vec{u} = \langle a, b, c \rangle$ $|\vec{u}| = 1$.

A contour (or level) surface of F is given by the points (x,y,z) $F(x,y,z) = k$ (constant). A similar argument to the one above shows that $\vec{\nabla} F(x_0, y_0, z_0)$ is orthogonal to the contour surface containing (x_0, y_0, z_0)

Thus we define the tangent plane to the contour surface to be the plane with normal $\vec{\nabla} F(x_0, y_0, z_0)$ and containing (x_0, y_0, z_0), namely:

$$\frac{\partial F}{\partial x}(x_0, y_0, z_0)(x - x_0) + \frac{\partial F}{\partial y}(x_0, y_0, z_0)(y - y_0) + \frac{\partial F}{\partial z}(x_0, y_0, z_0)(z - z_0) = 0$$

special case : surface $z = f(x,y)$ (11)

regard as contour-surface $F(x,y,z) = -z + f(x,y) = 0$

In this case, $\frac{\partial F}{\partial x}(x_0,y_0,z_0) = \frac{\partial f}{\partial x}(x_0,y_0)$

$$\frac{\partial F}{\partial y}(x_0,y_0,z_0) = \frac{\partial f}{\partial y}(x_0,y_0)$$

$$\frac{\partial F}{\partial z}(x_0,y_0,z_0) = -1$$

and the tangent plane is

$$\frac{\partial f}{\partial x}(x_0,y_0)(x-x_0) + \frac{\partial f}{\partial y}(y-y_0) - (z-z_0) = 0 \text{, as before.}$$

Example : Show that every tangent plane to the surface $x^2+y^2=z^2$ passes through the origin.

$\Rightarrow$ Find tangent plane at (x_0,y_0,z_0) : Le $F(x,y,z) = x^2+y^2-z^2$

Surface is $F(x,y,z)=0$