Gradient and Directional derivatives of functions of 3-variables

Similar ideas apply for 3 or more variables.

For $w = F(x,y,z)$ we define $\vec{\nabla} F = \left(\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z}\right)$ and the directional derivative is $D_{\vec{u}} F = \vec{\nabla} F \cdot \vec{u}$ with $\vec{u} = \langle a, b, c \rangle$ $|\vec{u}| = 1$.

A contour (or level) surface of F is given by the points (x,y,z) $F(x,y,z) = k$ (constant). A similar argument to the one above shows that $\vec{\nabla} F(x_0, y_0, z_0)$ is orthogonal to the contour surface containing (x_0, y_0, z_0)

Thus we define the tangent plane to the contour surface to be the plane with normal $\vec{\nabla} F(x_0, y_0, z_0)$ and containing (x_0, y_0, z_0), namely:

$$\frac{\partial F}{\partial x}(x_0,y_0,z_0)(x-x_0) + \frac{\partial F}{\partial y}(x_0,y_0,z_0)(y-y_0) + \frac{\partial F}{\partial z}(x_0,y_0,z_0)(z-z_0) = 0$$

Special case: surface $z = f(x,y)$

regard as contour-surface $F(x,y,z) = -z + f(x,y) = 0$

In this case, $\dfrac{\partial F}{\partial x}(x_0,y_0,z_0) = \dfrac{\partial f}{\partial x}(x_0,y_0)$

$$\frac{\partial F}{\partial y}(x_0,y_0,z_0) = \frac{\partial f}{\partial y}(x_0,y_0)$$

$$\frac{\partial F}{\partial z}(x_0,y_0,z_0) = -1$$

and the tangent plane is

$$\frac{\partial f}{\partial x}(x_0,y_0)(x-x_0) + \frac{\partial f}{\partial y}(y-y_0) - (z-z_0) = 0, \text{ as before}$$

<u>Example</u>: Show that every tangent plane to the surface $x^2+y^2=z^2$ passes through the origin.

$\Rightarrow$ Find tangent plane at (x_0,y_0,z_0): Le $F(x,y,z) = x^2+y^2-z^2$

Surface is $F(x,y,z)=0$.

Then $\dfrac{\partial F}{\partial x} = 2x$, $\dfrac{\partial F}{\partial y} = 2y$, $\dfrac{\partial F}{\partial z} = -2z$.

Tangent plane:

$$2x_0(x-x_0) + 2y_0(y-y_0) - 2z_0(z-z_0) = 0$$

Contains $(0,0,0)$? Yes if

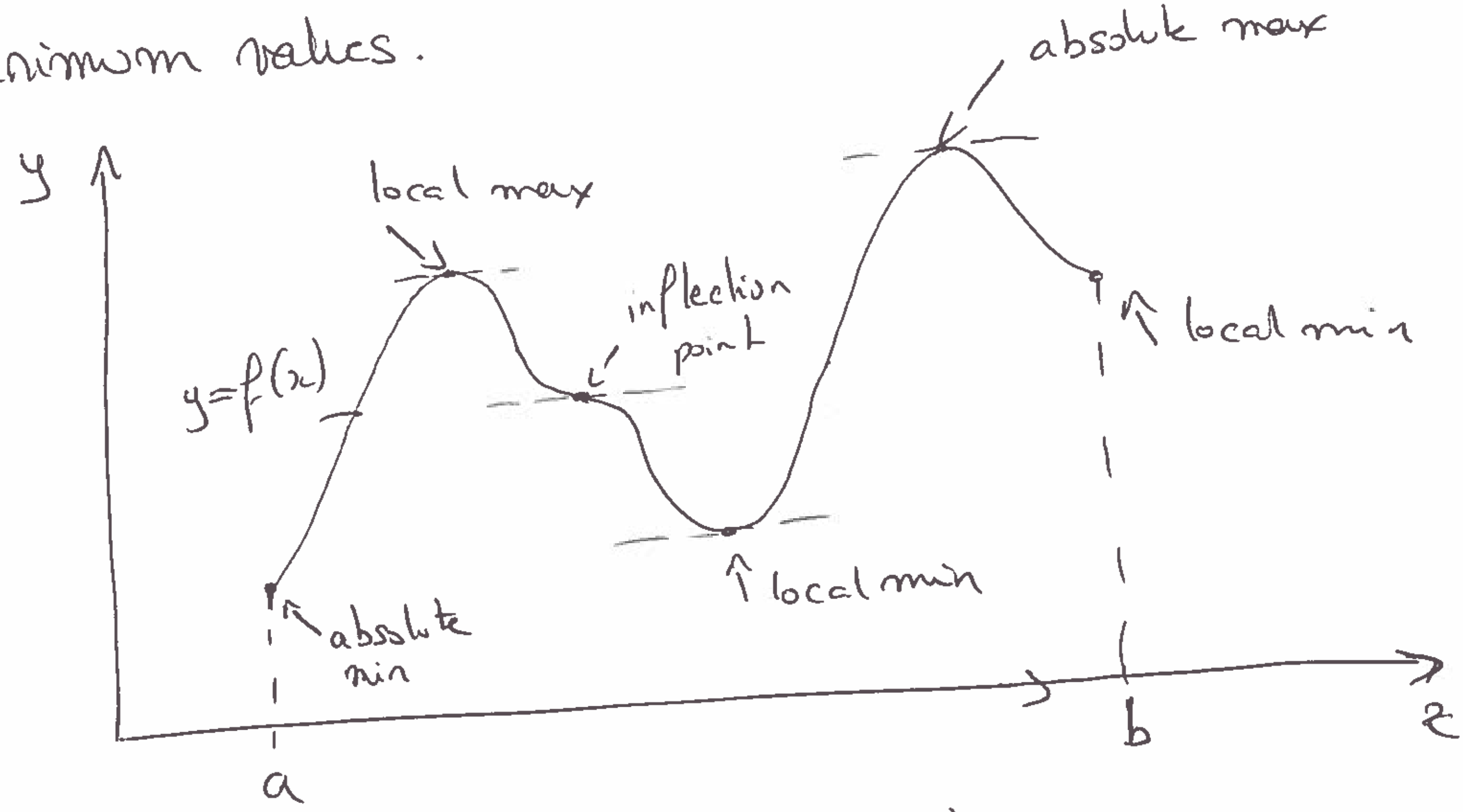
$$2x_0(-x_0) + 2y_0(-y_0) - 2z_0(-z_0) = 0$$

$$-2(\underbrace{x_0^2 + y_0^2 - z_0^2}_{=0}) = 0.$$ So Yes!

(because the equation of the surface is $x^2+y^2=z^2$)

Max / Min of functions of 2 variables

One of the most powerful ways to use calculus of one variable was to find maximum or minimum values.



to maximize or minimize $f(x)$ on an interval $[a,b]$, we look at

(*) critical points $f'(x)=0$

(*) boundary points

Critical points can be

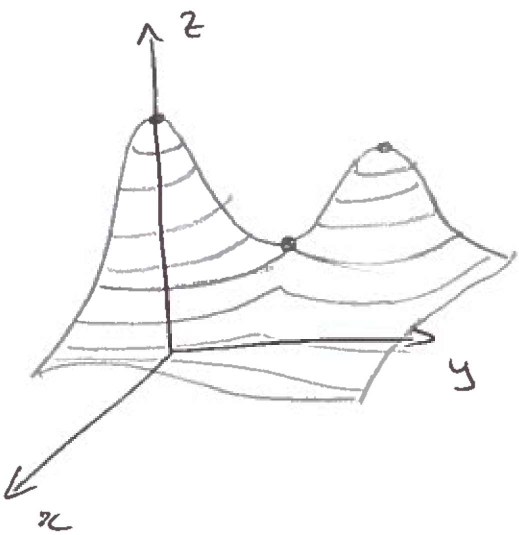
① local max $f''<0$

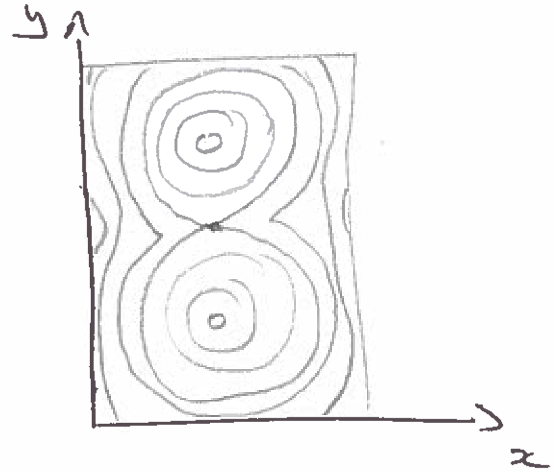
② local min $f''>0$

③ $f''=0$ could be either or neither.

For $f(x,y)$ on a closed and bounded domain R we will have the same strategy: critical points are the zeros of the partial derivatives.

Critical points:



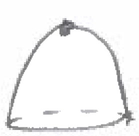


(x_0, y_0) is a critical point if $\vec{\nabla} f(x_0,y_0) = \vec{0}$

i.e: the tangent plane at (x_0,y_0) is horizontal.

ordinary critical points come in 3-kinds:

local max, local min, saddle point



hill

$z = -x^2 - y^2$

hole

$z = x^2 + y^2$

pass or saddle

$z = -x^2 + y^2$

Second derivative test (Proof is p. 930 in the Stewart, have a look it is interessting)

Let $\Delta(x,y) = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = f_{xx} f_{yy} - f_{xy}^2$

if (x_0, y_0) is a critical point then:

$\Delta(x_0,y_0) < 0 \Rightarrow (x_0,y_0)$ is a saddle.

$\Delta(x_0,y_0) > 0 \Rightarrow (x_0,y_0)$ is a local min or local max.

$\Delta(x_0,y_0) > 0$ and $f_{xx}(x_0,y_0) < 0$: local max.

$\Delta(x_0,y_0) \cancel{<} 0$ and $f_{xx}(x_0,y_0) > 0$: local min.

($\Delta(x_0,y_0) > 0$ and $f_{xx}(x_0,y_0) = 0$ is impossible, because $-f_{xy}^2$ cannot be positive)

exemple : Find and classify the critical points of $f(x,y) = (2x - x^2)(2y - y^2)$

$\vec{\nabla} f = \vec{0} \Rightarrow \frac{\partial f}{\partial x} = 0 = \frac{\partial f}{\partial y}$

$\Rightarrow \frac{\partial f}{\partial x} = (2y - y^2)(2 - 2x) = 2(1-x)(2-y)y$

$\Rightarrow \frac{\partial f}{\partial y} = (2x - x^2)(y - 2y) = 2(1-y)(2-x)x$