derivative test (Proof is p. 930 in the Stewart, have a look it is interesting)

Lecture 19

$$D(x,y) = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = f_{xx} f_{yy} - f_{xy}^2$$

if (x_0, y_0) is a critical point then:

$D(x_0, y_0) < 0 \Rightarrow (x_0, y_0)$ is a saddle.

$D(x_0, y_0) > 0 \Rightarrow (x_0, y_0)$ is a local min or local max.

$D(x_0, y_0) > 0$ and $f_{xx}(x_0, y_0) < 0$: local max.

$D(x_0, y_0) \cancel{<} 0$ and $f_{xx}(x_0, y_0) > 0$: local min.

$(D(x_0,y_0) > 0$ and $f_{xx}(x_0,y_0)) = 0$ is impossible, because $-f_{xy}^2$ cannot be positive,

example: Find and classify the critical points of $f(x,y) = (2x - x^2)(2y - y^2)$

$$\vec{\nabla} f = \vec{0} \Rightarrow \frac{\partial f}{\partial x} = 0 = \frac{\partial f}{\partial y}$$

$$\Rightarrow \frac{\partial f}{\partial x} = (2y - y^2)(2 - 2x) = 2(1-x)(2-y)y$$

$$\Rightarrow \frac{\partial f}{\partial y} = (2x - x^2)(y - 2y) = 2(1-y)(2-x)x$$

$$\frac{\partial f}{\partial x} = 0 \Rightarrow x = 1 \text{ or } 2 = y \text{ or } y = 0$$

$$\frac{\partial f}{\partial y} = 0 \Rightarrow y = 1 \text{ or } x = 2 \text{ or } x = 0$$

$x = 1$: $y = 1$ ✓, $\cancel{x = 2}$, $\cancel{x = 0}$

$y = 2$: $\cancel{y = 1}$, $x = 2$ ✓, $x = 0$ ✓

$y = 0$: $y = 1$, $x = 2$ ✓, $x = 0$ ✓

so $(1,1)$, $(2,2)$, $(2,0)$, $(0,2)$, $(0,0)$ are critical points.

$$\frac{\partial^2 f}{\partial x^2} = -2(2-y)y \qquad \frac{\partial^2 f}{\partial xy} = 2(1-y)(2-2x)$$

$$\frac{\partial^2 f}{\partial y^2} = -2(2x - x^2)$$

$\Rightarrow$ it comes

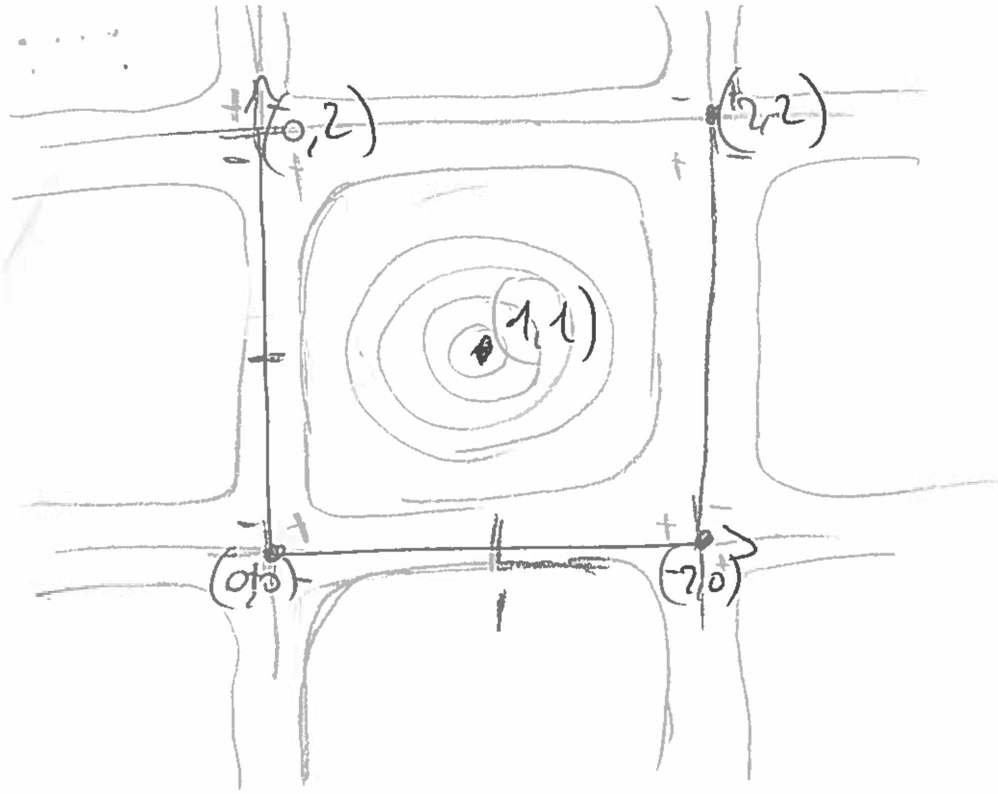
$\Delta(1,1) = 4$ and $\frac{\partial^2 f}{\partial x^2}(1,1) = -2 \Rightarrow (1,1)$ is a local max.

$\Delta(2,2) = -16 \Rightarrow (2,2)$ is a saddle

$\Delta(2,0) = -16 \Rightarrow (2,0)$ is "

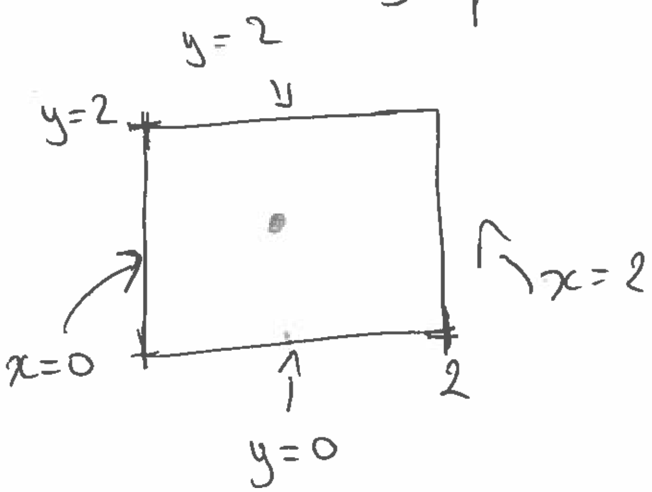
$\Delta(0,2) = -16 \Rightarrow (0,2)$ is "

$\Delta(0,0) = -16 \Rightarrow (0,0)$ is "



<u>Example</u>: find absolute max and mins on the domain $R = \{0 \leq x \leq 2,\ 0 \leq y \leq 2\}$

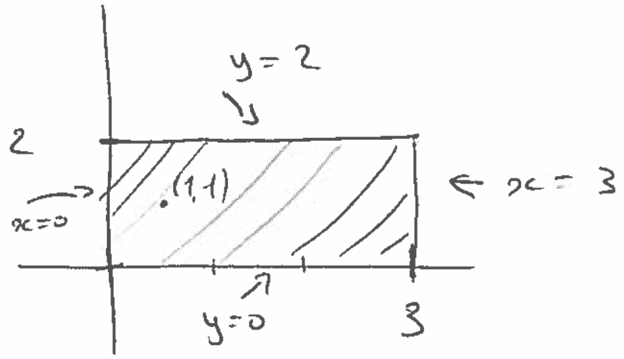
$\Rightarrow$ Boundary of R is the 4 segments



$f = 0$ on the boundaries (abs min)

$f(1,1) = 1$ abs max.

how about $R = \{ 0 \leq x \leq 3,\ 0 \leq y \leq 2 \}$?



$\Rightarrow$ function is 0 on the top, bottom and left boundary. On the $x = 3$ boundary:

$f(x,y)$ is $f(3,y)$ for $0 \leq y \leq 2$

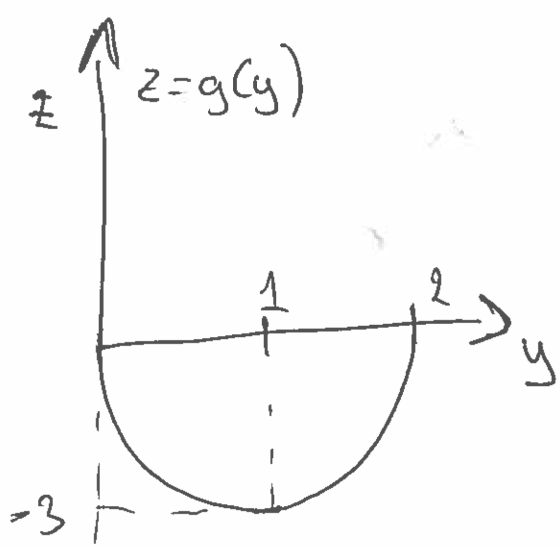
$\Rightarrow f(3,y) = -3(2y - y^2) = -6y + 3y^2$

$\Rightarrow$ find extrema of $z = g(y) = -6y + 3y^2 \quad 0 \leq y \leq 2$

$\Rightarrow g'(y) = 0 = -6 + 6y \Rightarrow y = 1$

$\Rightarrow g''(y) = 6 > 0 \Rightarrow$ local min with $g(1) = -3$

(concave ~~up~~)



Therefore on R $(1,1)$ is abs max with $f(1,1) = 1$ and $(3,1)$ is a ~~abs~~ min with $f(3,1) = -3$