Lecture 23

Problem: Find the average value of $f(x,y) = e^y\sqrt{x+e^y}$ on the rectangle $(x,y) \in [0,4]\times[0,1]$

$$\Rightarrow f_{average} = \frac{1}{(4-0)}\frac{1}{(1-0)}\int_0^{x=4}\left[\int_{y=0}^{y=1} e^y\sqrt{x+e^y}\,dy\right]dx$$

$$= \frac{1}{4}\int_{x=0}^{x=4}\left[\int_{u=x+1}^{x+e}\sqrt{u}\;du\right]dx$$

$u = x+e^y$

$du = e^y\,dy$

(treat x as constant here)

$$= \frac{1}{4}\int_{x=0}^{x=4}\left[\frac{2}{3}u^{3/2}\right]_{x+1}^{x+e}dx$$

$$= \frac{1}{6}\int_{x=0}^{x=4}(x+e)^{3/2} - (x+1)^{3/2}\;dx$$

$$= \frac{1}{6}\left[\frac{2}{5}(x+e)^{5/2} - \frac{2}{5}(x+1)^{5/2}\right]_{x=0}^{x=4}$$

②

$$= \frac{1}{15}\left[(4+e)^{5/2} - (4+1)^{5/2} - \left(e^{5/2} - (1)^{5/2}\right)\right]$$

$$= \frac{1}{15}\left[(4+e)^{5/2} - 5^{5/2} - e^{5/2} + 1\right] \approx 3,327$$

example: Evaluate $I = \iint_R x^y \, dx dy \quad R = [0,1]\times[0,1]$

=> Use the Fubini theorem, one order can be easier than the other!

=> method 1:

$$\int_0^1 \left[\int_0^1 x^y \, dx\right] dy = \int_0^1 \left[\frac{x^{y+1}}{y+1}\right]_0^1 dy = \int_0^1 \frac{1}{y+1} \, dy$$

$$= \left[\ln(y+1)\right]_0^1 = \ln(2) - \underbrace{\ln(1)}_{=0} = \ln(2)$$

=> method 2

$$\int_0^1 \left[\int_0^1 x^y \, dy\right] dx = \int_0^1 \int_0^1 e^{y \ln x} \, dy \, dx = \int_0^1 \left[\frac{e^{y \ln x}}{\ln x}\right]_0^1 dx$$

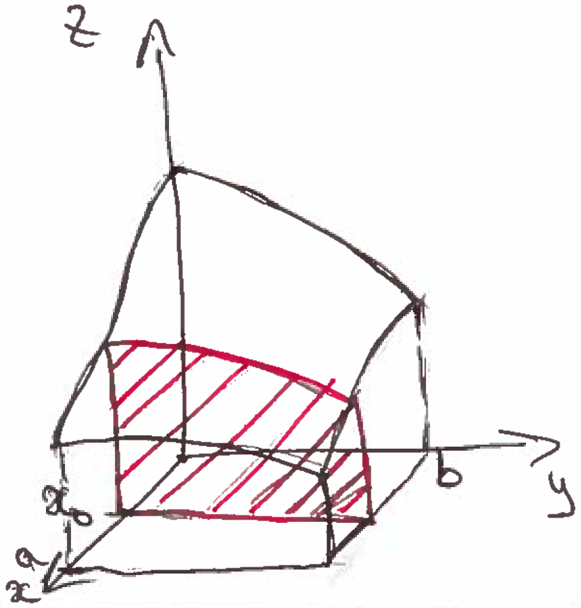
$$= \int_0^1 \frac{e^{\ln x} - 1}{\ln x}\,dx = \int_0^1 \frac{x-1}{\ln x}\,dx$$ outch! :(

$\Rightarrow$ Side benefit: $\displaystyle\int_0^1 \frac{x-1}{\ln x}\,dx = \ln(2)$. :)

$\Rightarrow$ Double Integrals over general Regions

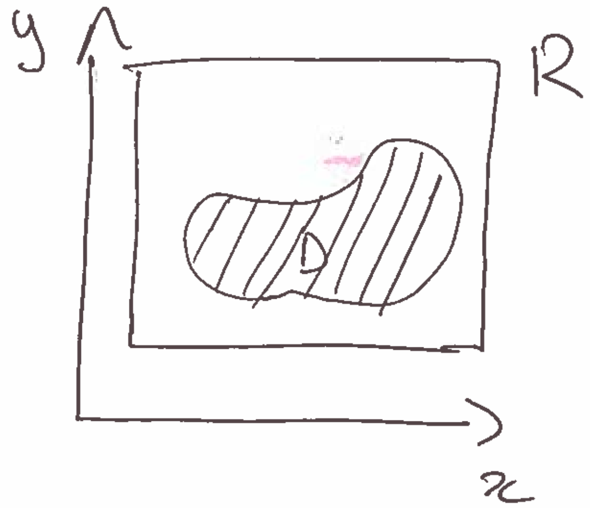
Recall: $\displaystyle\iint_R f(x,y)\,dx\,dy$ = volume under graph

$$= \int_{x=0}^{x=a} \underbrace{\left[\int_{y=0}^{y=b} f(x,y)\,dy\right]}_{\substack{g(x) = \text{area under curve} \\ \text{with } x \text{ fixed}}} dx$$

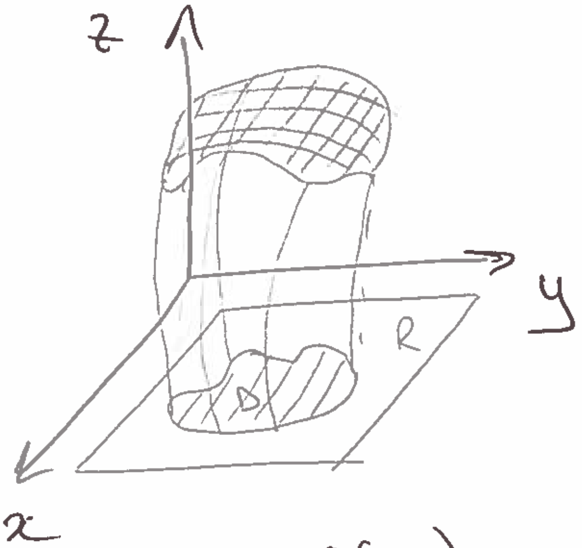


$g(x_0)$ area under the trace curve $z = f(x_0, y)$

$\Rightarrow$ what if we want to integrate over a more general domain? Say closed and bounded?



$$F(x,y) = \begin{cases} f(x,y) & \text{if } (x,y) \in D \\ 0 & \text{if } (x,y) \in R - D \end{cases}$$



definition:

$$\iint_D f(x,y)\,dx\,dy = \iint_R F(x,y)\,dx\,dy$$

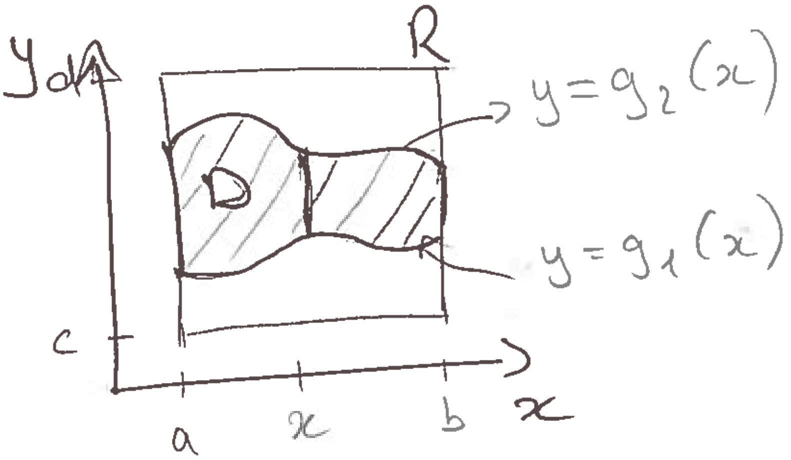
graph of $z = F(x,y)$ $\leftarrow$ not continuous but $\iint_R F(x,y)\,dx\,dy$ exists if boundary of D is not too wild.

Interpretation: $\iint_D f(x,y)\,dx\,dy$ = volume of solid under graph of f and over D.

How to evaluate $\iint_D f(x,y)\,dx\,dy$?

$\Rightarrow$ it depends on form of region D.

"Type I" region: $D=\{(x,y): a\leq x\leq b,\ g_1(x)\leq y\leq g_2(x)\}$



by definition:

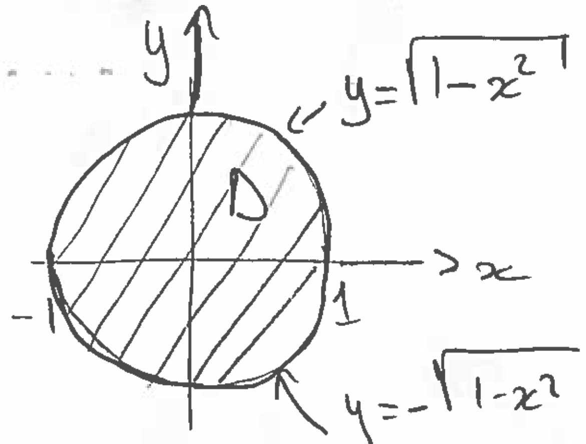
$$\iint_D f(x,y)\,dx\,dy = \iint_R F(x,y)\,dx\,dy$$

$$= \int_a^b \int_c^d F(x,y)\,dy\,dx$$

$$= \int_a^b \left[\int_{y=g_1(x)}^{y=g_2(x)} f(x,y)\,dy \right] dx$$

example: evaluate $I = \iint_D f(x,y)\,dx\,dy$ if $f(x,y) = (1-x^2)$

and $D = \{(x,y) : x^2+y^2 \leq 1\}$



D is of type I

$$D=\{(x,y):-1\leq x\leq 1,-\sqrt{1-x^2}\leq y\leq \sqrt{1-x^2}\}$$

$$\Rightarrow I=\int_{x=-1}^{x=1}\int_{y=-\sqrt{1-x^2}}^{y=\sqrt{1-x^2}}(1-x^2)^{3/2}\,dy\,dx=\int_{x=-1}^{x=1}(1-x^2)^{3/2}\int_{y=-\sqrt{1-x^2}}^{y=+\sqrt{1-x^2}}dy\,dx$$

$$=\int_{x=1}^{x=1}(1-x^2)^{3/2}\Big[y\Big]_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}dx=\int_{x=-1}^{x=1}(1-x^2)^{3/2}\left(2\sqrt{1-x^2}\right)dx$$

$$=\int_{x=-1}^{x=1}2(1-x^2)^{2}\,dx=2\int_{x=-1}^{x=1}(1-2x^2+x^4)\,dx$$

$$=2\left[x-\frac{2x^3}{3}+\frac{x^5}{5}\right]_{-1}^{+1}=2\left[\left(1-\frac{2}{3}+\frac{1}{5}\right)-\left(-1+\frac{2}{3}-\frac{1}{5}\right)\right]$$

$$I=4-\frac{8}{3}+\frac{4}{5}=\frac{60-40+12}{15}=\frac{32}{15}$$