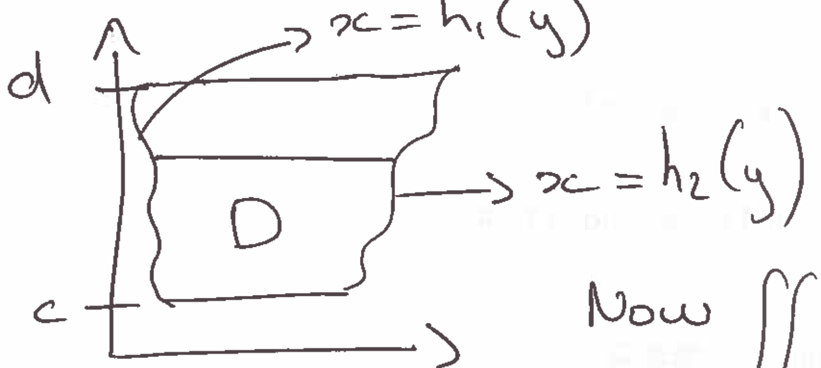
Lecture 24

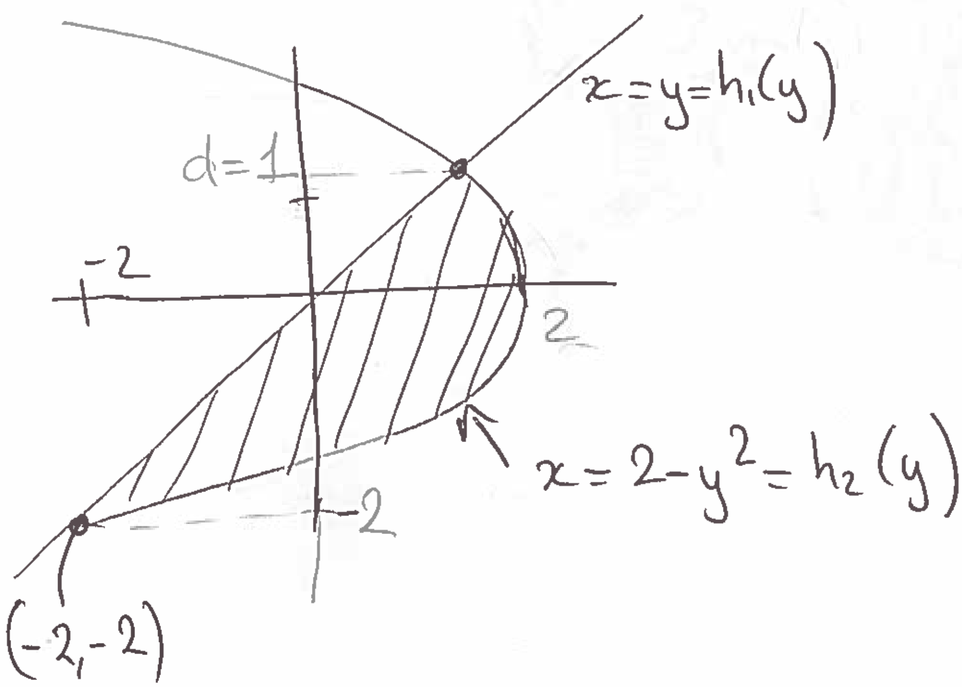
Types II regions.

$$D = \{(x,y) : c \leq y \leq d ,\ h_1(y) \leq x \leq h_2(x)\}$$



Now $\displaystyle\iint_D f(x,y)\,dx\,dy = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\,dx\,dy$

Example: Let D be the region bounded by $y = x$ and $x = 2-y^2$. Find $\displaystyle\iint_D y\,dx\,dy$



$\Rightarrow$ need to figure out y end points, where do the curves intersect?

$$y = 2 - y^2 \Rightarrow y^2 + y - 2 = 0 \Rightarrow (y-1)(y+2) = 0$$

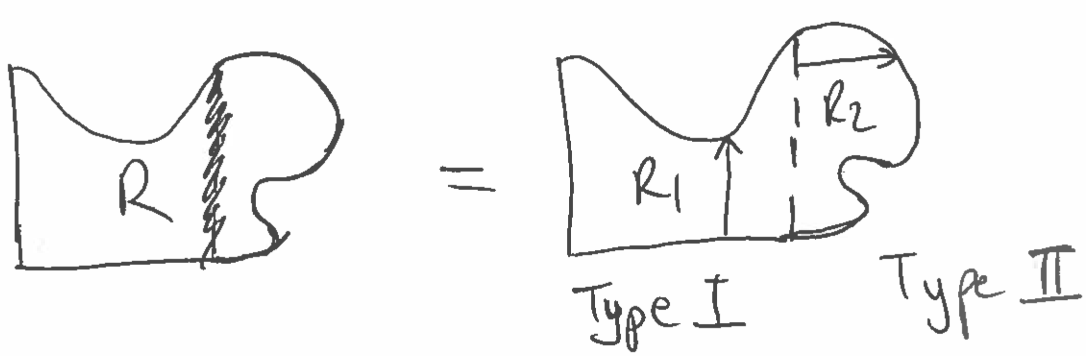
$$y = -2 \text{ or } 1$$

$$\iint_D y \, dy\, dx = \int_{y=-2}^{y=1} \int_{x=y}^{x=2-y^2} y \, dx\, dy = \int_{y=-2}^{y=1} (2-y^2-y)\, y \, dy$$

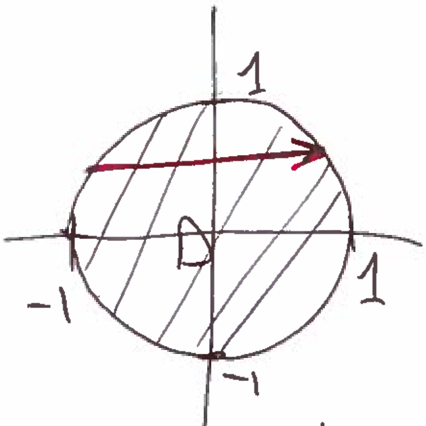
$$= \left[\frac{\cancel{2} y^2}{\cancel{2}} - \frac{y^4}{4} - \frac{y^3}{3} \right]_{-2}^{1} = \left[\left(1 - \frac{1}{4} - \frac{1}{3}\right) - \left(4 - \frac{(-2)^4}{4} - \frac{(-2)^3}{3}\right) \right]$$

$$= 1 - \frac{1}{4} - \frac{1}{3} - \cancel{4} + \frac{\cancel{16}}{\cancel{4}} - \frac{8}{3} = -3 - \frac{1}{4} + 1 = \frac{-9}{4}$$

$$\iint_R f(x,y)\, dx dy = \iint_{R_1} f(x,y)\, dx dy + \iint_{R_2} f(x,y)\, dx dy$$



Example : $I_2 = \iint_D (1-x^2)^{3/2}\,dx\,dy$, again



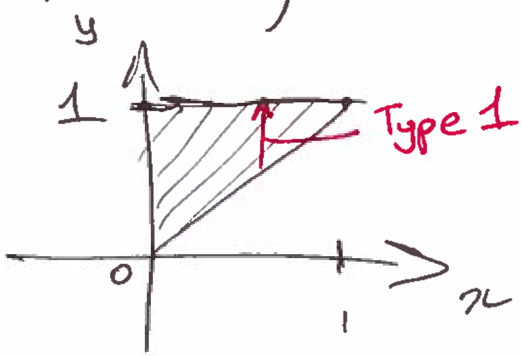
This time regard as type II region: (integrate in x direction first)

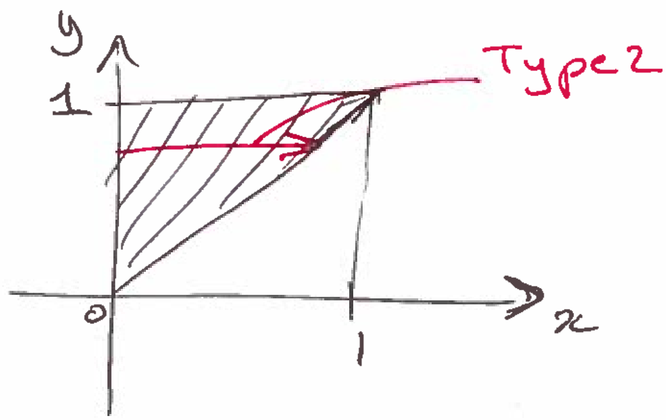
$$I_2 = \int_{y=-1}^{y=1} \underbrace{\int_{x=-\sqrt{1-y^2}}^{x=+\sqrt{1-y^2}} (1-x^2)^{3/2}\,dx}\,dy$$

Nasty, needs trig substitution etc...
$\Rightarrow$ Leads to a big mess
$\Rightarrow$ Better to use Type 1.

Example : Evaluate : $I = \int_0^1 \int_x^1 \sin(y^2)\,dy\,dx$

Difficulty : $\int \sin(y^2)\,dy$ is not an elementary function.





so! switch types

$$I = \int_0^1 \underbrace{\int_0^y \sin(y^2)\,dx}_{\sin(y^2)\int_0^y dx = y\sin(y^2)}\,dy = \int_0^1 y\sin(y^2)\,dy$$

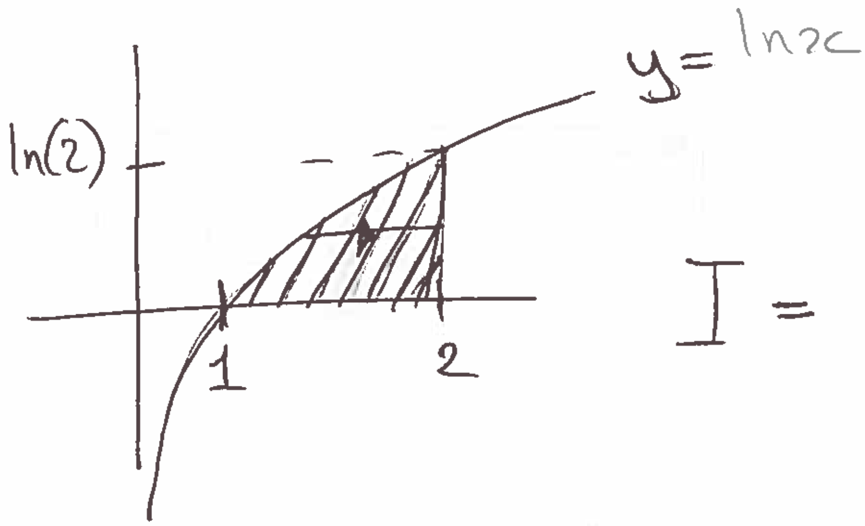
$$I = \int_{0^2}^{1^2} \sin(u)\frac{1}{2}du \qquad \text{with } u = y^2 \text{ and } du = 2y\,dy$$

$$I = \frac{1}{2}\left[-\cos u\right]_0^1 = \cancel{\frac{1}{2}(-1-1) = -1}$$

$$= \frac{1}{2}\left[-\left(\cos(1) - \cos(0)\right)\right] = \frac{1}{2}\left(1 - \cos 1\right) = 0{,}2298..$$

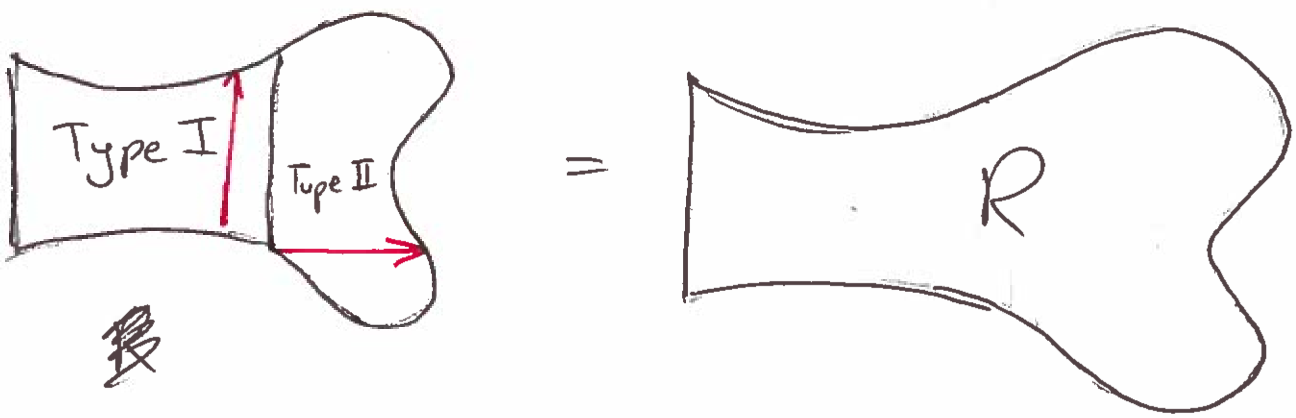
Example: Reverse the order of integration (4)

for $I = \int_1^2 \int_0^{\ln x} f(x,y)\, dy\, dx$



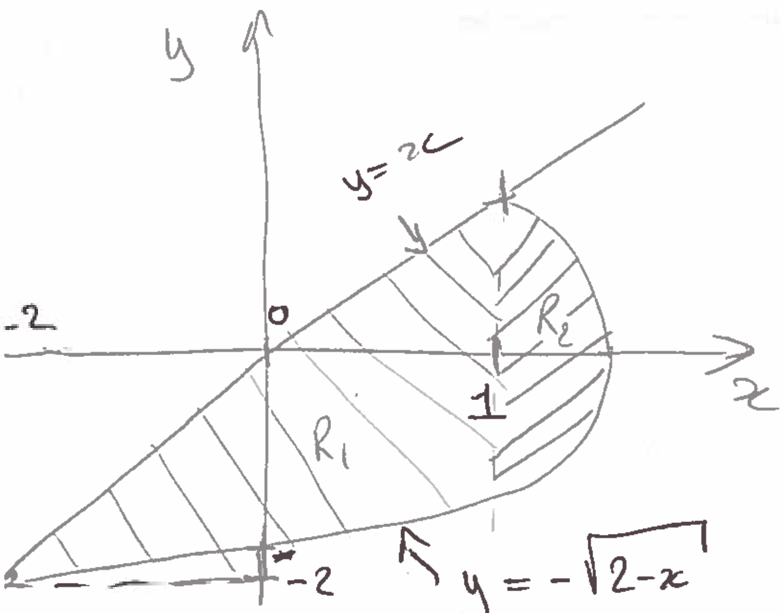
$$I = \int_0^{\ln(2)} \int_{e^y}^{2} f(x,y)\, dx\, dy$$

More general domains.



$$\iint_R f(x,y)\,dx\,dy = \iint_{R_1} f(x,y)\,dx\,dy + \iint_{R_2} f(x,y)\,dx\,dy$$

Previous example



$$\iint_R y\,dy\,dx = \iint_{R_1} y\,dx\,dy + \underbrace{\iint_{R_2} y\,dx\,dy}_{=0}$$

why?

$$\int_{x=-2}^{x=1}\int_{-\sqrt{2-x}}^{x} y\,dy\,dx = \int_{x=-2}^{x=1}\left[\frac{y^2}{2}\right]_{-\sqrt{2-x}}^{x} dx = \frac{1}{2}\int_{x=-2}^{x=1} x^2 - \sqrt{2-x}^{\,2}\,dx$$

$$= \frac{1}{2}\int_{-2}^{1} x^2 - 2 + x\,dx = \frac{1}{2}\left[\frac{x^3}{3} - 2x + \frac{x^2}{2}\right]_{-2}^{1}$$

$$= \frac{1}{2}\left[\frac{1}{3} - 2 + \frac{1}{2} - \left(\frac{-8}{3} + 4 + \frac{4}{2}\right)\right] = \frac{1}{6} - 1 + \frac{1}{4} + \frac{8}{6} - 2 - 1$$

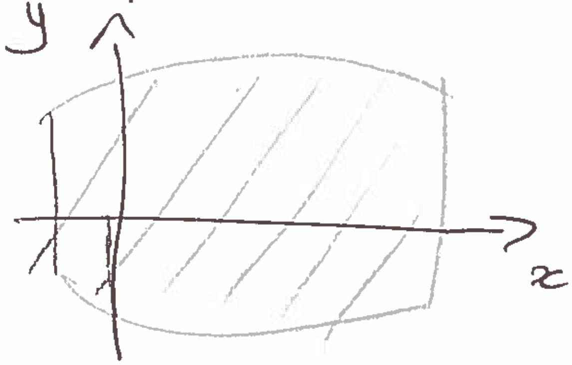
$$= \frac{9}{6} - 4 + \frac{1}{4} = \frac{3}{2} - 4 + \frac{1}{4} = \frac{6}{4} - \frac{16}{4} + \frac{1}{4} = \frac{-9}{4}$$

ok

Symmetry can be useful:

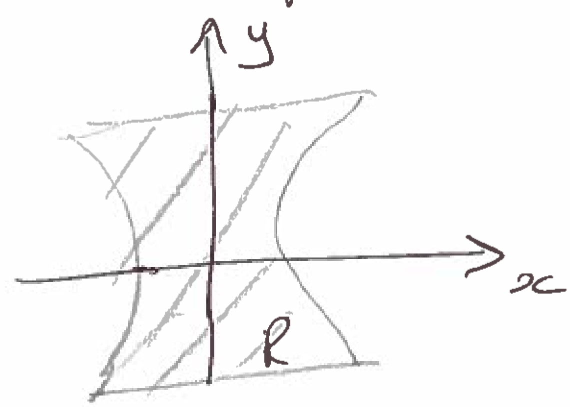
- if R is symmetric about x axis and $f(x,-y) = -f(x,y)$ (odd function of y)

  then $\displaystyle\iint_R f(x,y)\,dx\,dy = 0$



- if R is symmetric about y axis and $f(-x,y) = -f(x,y)$ (odd function of x)

  then $\displaystyle\iint_R f(x,y)\,dx\,dy = 0$



Example: let $R = \{(x,y) : x^2+y^2 \leq 4\}$

$$I = \iint_R \underbrace{e^{y} x^2 \tan x}_{\text{odd in } x} + \underbrace{\sin(xy^3)}_{\text{odd in } y} + 5 \; dx\,dy$$

$$I = 0 + 0 + \iint_R 5\, dx\,dy = 5 \times \text{area}(R) = 5\pi(2)^2$$
$$= 20\pi$$