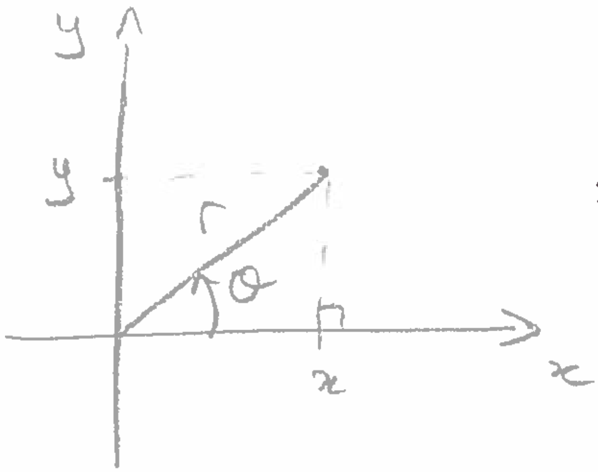
Lecture 25

Double integrals in Polar Coordinates

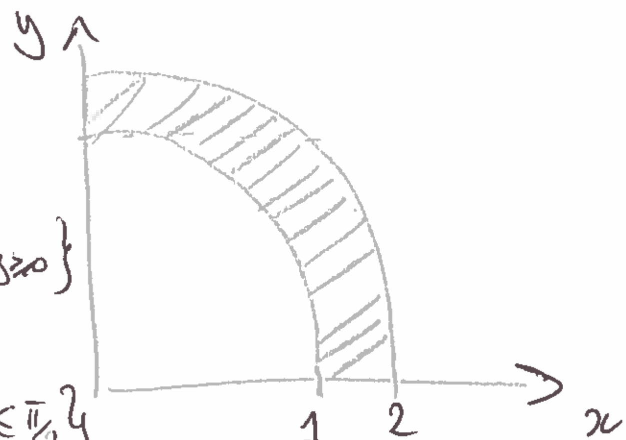


$$r^2 = x^2 + y^2$$

$$x = r\cos\theta$$

$$y = r\sin\theta$$

Often the region and/or the function is easier to write in polar coords. Especially true in scientific applications where problems often have a rotational symmetry. e.g

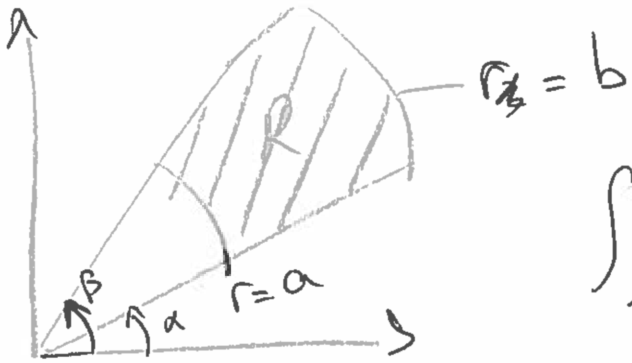


$R = \{(x,y) : 1 \leq x^2+y^2 \leq 2,\ x \geq 0, y \geq 0\}$

$R = \{(r,\theta) : 1 \leq r \leq \sqrt{2}\ \ 0 \leq \theta \leq \pi/2\}$

R is akward to do as type I or type II.

Definition: A "polar rectangle" is $R=\{(r,\theta)\, a\leq r\leq b, \alpha\leq\theta\leq\beta\}$



We want to consider $\iint_R f(x,y)\,dA$ where R is a polar rectangle.

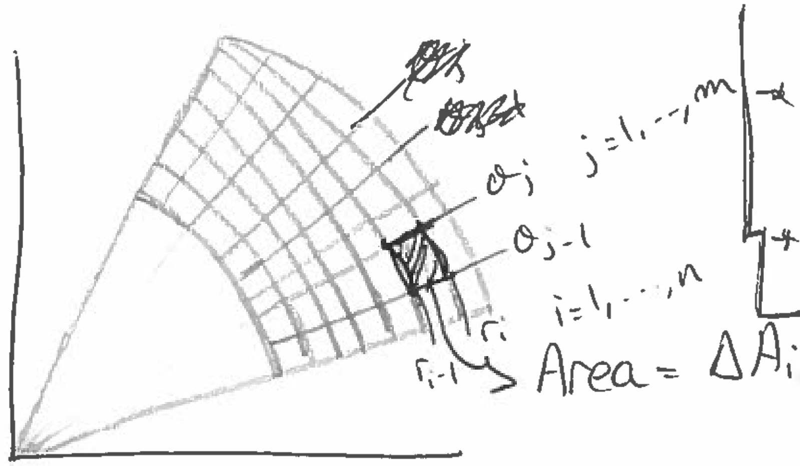
=> Recall: for ordinary Rectangle R_0, we defined

$$\iint_R f(x,y)\,dA = \lim_{\substack{m\to\infty \\ n\to\infty}} \sum_{i=1}^{n}\sum_{j=1}^{m} f(x_i,y_j)\underbrace{\Delta x\Delta y}_{\Delta A}$$

For polar rectangle R, we subdivide R into polar subrectangles.

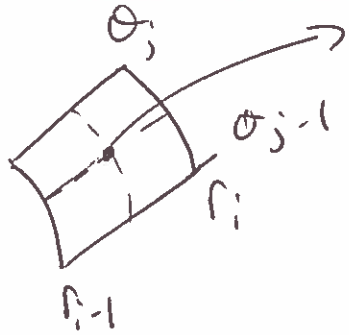
=> area of a slice of pie 

is $\frac{\theta}{2\pi}\pi r^2 = \frac{1}{2}\theta r^2$



* divide up $[a,b]$ into n pieces of size $\Delta r=\frac{b-a}{n}$
* divide up $[\alpha,\beta]$ into m pieces or size $\Delta\theta=\frac{\beta-\alpha}{m}$

Zoom on a polar sub-rectangle.



→ centre is (r_i^*, θ_j^*) where

$$r_i^* = \frac{1}{2}(r_i + r_{i-1})$$

$$\theta_j^* = \frac{1}{2}(\theta_j + \theta_{j-1})$$

In rectangular coords, centre is $(r_i^* \cos\theta_j^*, r_i^* \sin\theta_j^*)$

as area of slice of pie is  $\left(\frac{\theta}{2\pi}\right)\pi r^2 = \frac{\theta r^2}{2}$

the area of ΔA_i is $\frac{\Delta\theta\, r_i^2}{2} - \frac{\Delta\theta\, r_{i-1}^2}{2}$

$$\Rightarrow \Delta A_i = \frac{1}{2}(r_i^2 - r_{i-1}^2)\Delta\theta$$

$$\Delta A_i = \underbrace{\frac{1}{2}(r_i + r_{i-1})}_{r_i^*}\underbrace{(r_i - r_{i-1})}_{\Delta r}\Delta\theta$$

$$\Delta A_i = r_i^* \Delta r \Delta\theta$$

Polar Riemann sum: $\sum_{i=1}^{n}\sum_{j=1}^{m} f(r_i^*\cos\theta_j^*, r_i^*\sin\theta_j^*)\,\Delta A_i$

$$= \sum_{i=1}^{n}\sum_{j=1}^{m} \underbrace{f(r_i^*\cos\theta_j^*, r_i^*\sin\theta_j^*)\, r_i^*}_{\text{call this } g(r_i^*, \theta_j^*)}\Delta r\Delta\theta$$

Let $m, n \to \infty$ Riemann sum converges to $\displaystyle\int_{\alpha}^{\beta}\int_{a}^{b} g(r,\theta)\,dr\,d\theta$

and we found that:

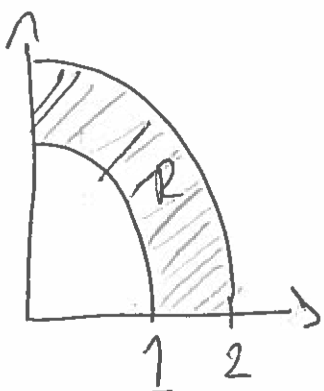
$$\iint_R f(x,y)\,dA = \text{limit of polar sum}$$

$$= \int_{\theta=\alpha}^{\theta=\beta}\int_{r=a}^{r=b} f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta$$

Think of $dA = r\,dr\,d\theta$

dA approximately a standard rectangle of area $r\,dr\,d\theta$

Example



Evaluate $I = \iint_R x^2\,dA$

$$I = \int_0^{\pi/2}\int_1^2 (r\cos\theta)^2 r\,dr\,d\theta = \int_0^{\pi/2}\int_1^2 r^3\cos^2\theta\,dr\,d\theta$$

$$= \int_0^{\pi/2}\left[\frac{r^4}{4}\cos^2\theta\right]_1^2 d\theta = \int_0^{\pi/2}\cos^2\theta\left(4 - \frac{1}{4}\right)d\theta$$

use $\cos^2\theta = \frac{1}{2}(1+\cos 2\theta)$

$$= \frac{15}{4}\int_0^{\pi/2}\cos^2\theta\,d\theta = \frac{15}{8}\int_0^{\pi/2}(1+\cos 2\theta)\,d\theta$$