Let $m, n \to \infty$ Riemann sum converges to $\displaystyle\int_{\alpha}^{\beta}\int_{a}^{b} g(r,\theta)\,dr\,d\theta$

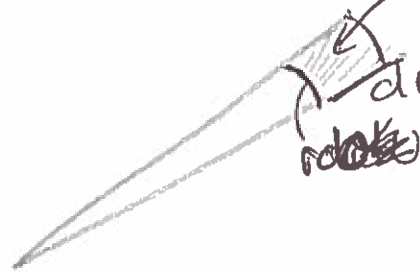
and we found that:

$$\iint_R f(x,y)\,dA = \text{limit of polar sum}$$

$$= \int_{\theta=\alpha}^{\theta=\beta}\int_{r=a}^{r=b} f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta$$

Think of $dA = r\,dr\,d\theta$:

dA approximately a standard rectangle of area $r\,dr\,d\theta$

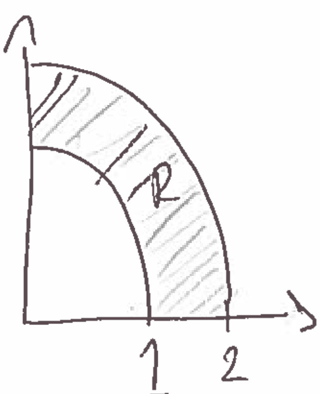


(θ doesn't have a dimension!)

Example

$\boxed{\text{Lecture 26}}$

Evaluate $I = \iint_R x^2\,dA$



$$I = \int_0^{\pi/2}\int_1^2 (r\cos\theta)^2\,r\,dr\,d\theta = \int_0^{\pi/2}\int_1^2 r^3\cos^2\theta\,dr\,d\theta$$

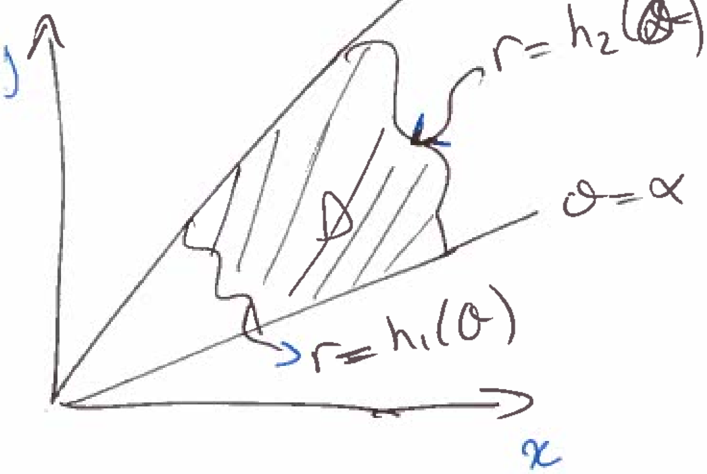
$$= \int_0^{\pi/2}\left[\frac{r^4}{4}\cos^2\theta\right]_1^2 d\theta = \int_0^{\pi/2}\cos^2\theta\left(4-\frac{1}{4}\right)d\theta$$

use $\cos^2\theta = \frac{1}{2}(1+\cos 2\theta)$

$$= \frac{15}{4}\int_0^{\pi/2}\cos^2\theta\,d\theta = \frac{15}{8}\int_0^{\pi/2}(1+\cos 2\theta)\,d\theta$$

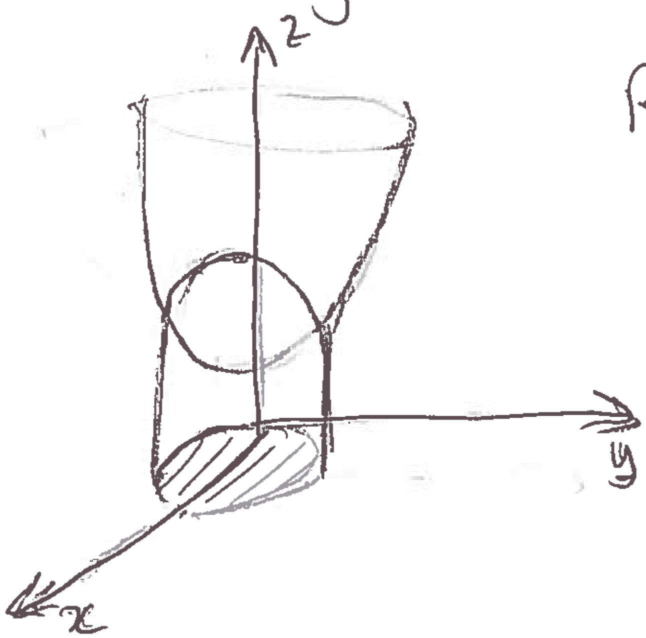
$$I = \frac{15}{8}\left[\theta + \frac{1}{2}\sin 2\theta\right]_0^{\pi/2} = \frac{15}{8}\left[\frac{\pi}{2}\right] = \frac{15\pi}{16}$$

we also have the analogue of Type I region in polar:

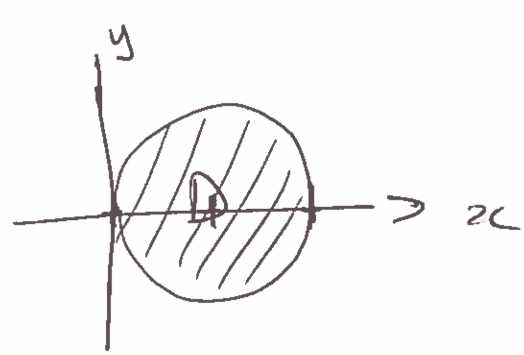


$$\iint_D f(x,y)\,dA = \int_{\theta=\alpha}^{\theta=\beta}\int_{r=h_1(\theta)}^{r=h_2(\theta)} f(r\cos\theta, r\sin\theta)\, r\,dr\,d\theta$$

example: Find the volume lying below the paraboloid $z = x^2 + y^2$ above the xy plane and inside the cylinder $(x-1)^2 + y^2 = 1$



$$R = \{(x,y) : (x-1)^2 + y^2 \leq 1\}$$



The boundary of D is $r = 2\cos\theta$:

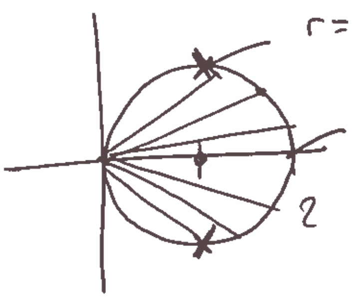
$$\Rightarrow (x-1)^2 + y^2 = 1 = x^2 - 2x + 1 + y^2 = 1$$

$$\cancel{r}^2\cos^2\theta - 2r\cos\theta + \cancel{1} + r^2\sin^2\theta = \cancel{1}$$

$$r^2\underbrace{(\cos^2\theta + \sin^2\theta)}_{=1} = 2r\cos\theta$$

$$\Rightarrow \boxed{r = 2\cos\theta}$$

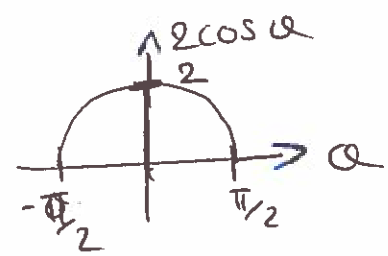
$$D = \left\{(r,\theta) : 0 \le r \le 2\cos\theta,\ -\tfrac{\pi}{2} \le \theta \le \tfrac{\pi}{2}\right\}$$



$r = 1$, $\theta = \pi/3$, $\theta = -\pi/3$ as $2\cos(\pi/3) = 2 \times \frac{1}{2} = 1$

$r = 2$, $\theta = 0$

$r = 0$, $\theta = \pi/2$ or $-\frac{\pi}{2}$



$$\Rightarrow \text{Volume } V = \iint_D (x^2+y^2)\,dA = \int_{-\pi/2}^{\pi/2}\int_0^{2\cos\theta} r^2\, r\,dr\,d\theta = \int_{-\pi/2}^{\pi/2} \frac{1}{4}(2\cos\theta)^4\,d\theta$$

$$V = \int_{-\frac{\pi}{2}}^{\pi/2} 4\cos^4\theta\,d\theta = 4 \times 2\int_0^{\pi/2} \cos^4\theta\,d\theta$$

$$\cos^4\theta = (\cos^2\theta)^2 = \left[\frac{1}{2}(1+2\cos\theta)\right]^2 = \frac{1}{4}(1+2\cos 2\theta + \underbrace{\cos^2 2\theta}_{\frac{1}{2}(1+\cos 4\theta)})$$ 7

$$\cos^4\theta = \frac{1}{4}\left(1+2\cos 2\theta + \frac{1}{2}(1+\cos 4\theta)\right)$$

$$\cos^4\theta = \frac{1}{4} + \frac{1}{2}\cos 2\theta + \frac{1}{8} + \frac{1}{8}\cos 4\theta$$

$$V = 8\left[\frac{1}{4}\theta + \frac{1}{4}\sin 2\theta + \frac{1}{8}\theta + \frac{1}{32}\sin 4\theta\right]_0^{\pi/2} = \frac{3\pi}{2}$$

Example : The standard bell curve : $f(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$

Evaluate $I = \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\, dx$

Difficulty $\Rightarrow$ $\int_{-\infty}^{+\infty} e^{-x^2/2}$ is not an elementary function!

$\Rightarrow$ Cool trick :

$$I^2 = \int_{-\infty}^{+\infty} e^{-x^2/2}\, dx \int_{-\infty}^{+\infty} e^{-y^2/2}\, dy = \iint_{\mathbb{R}^2} e^{-\frac{x^2}{2}-\frac{y^2}{2}}\, dx\, dy$$