$$\cos^4\theta = (\cos^2\theta)^2 = \left[\frac{1}{2}(1+2\cos\theta)\right]^2 = \frac{1}{4}(1+2\cos 2\theta + \underbrace{\cos^2 2\theta}_{\frac{1}{2}(1+\cos 4\theta)})$$ ⑦

$$\cos^4\theta = \frac{1}{4}\left(1+2\cos 2\theta + \frac{1}{2}(1+\cos 4\theta)\right)$$

$$\cos^4\theta = \frac{1}{4} + \frac{1}{2}\cos 2\theta + \frac{1}{8} + \frac{1}{8}\cos 4\theta$$

$$V = 8\left[\frac{1}{4}\theta + \frac{1}{4}\sin 2\theta + \frac{1}{8}\theta + \frac{1}{32}\sin 4\theta\right]_0^{\pi/2} = \frac{3\pi}{2}$$

$\boxed{\text{Lecture 27}}$

Example: The standard bell curve: $f(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$

Evaluate $I = \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\,dx$

Difficulty $\Rightarrow \int_{-\infty}^{+\infty} e^{-x^2/2}$ is not an elementary function!

$\Rightarrow$ Cool trick:

$$I^2 = \int_{-\infty}^{+\infty} e^{-x^2/2}\,dx \int_{-\infty}^{+\infty} e^{-y^2/2}\,dy = \iint_{\mathbb{R}^2} e^{-\frac{x^2}{2}-\frac{y^2}{2}}\,dx\,dy$$

$$I^2 = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \frac{1}{2\pi} e^{-(x^2+y^2)/2} \, dx\,dy = \frac{1}{2\pi}\int_0^{\infty}\int_0^{2\pi} e^{-r^2/2} \, r\,d\theta\,dr$$

$$= \frac{1}{\cancel{2\pi}}\int_0^{\infty} \cancel{2\pi} r e^{-r^2/2} \, dr = \int_0^{\infty} r e^{-r^2/2} \, dr$$

$u = \frac{r^2}{2}$

$du = r\,dr$

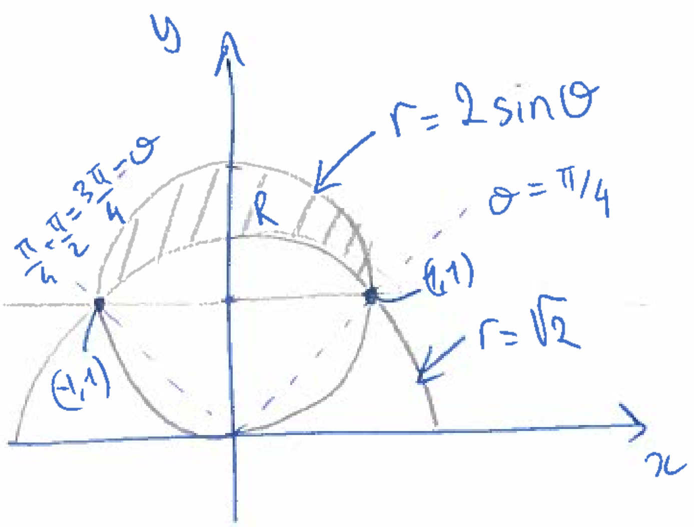
$u(0) \Rightarrow 0$

$u(\infty) \Rightarrow \infty$

$$= \int_0^{\infty} + e^{-u} \, du = -e^{-u}\Big|_0^{\infty} = \underline{1}$$

$\Rightarrow$ so $I = \sqrt{I^2} = \underline{1}$

Example: Find area of region outside circle of radius $\sqrt{2}$ centred at origin and inside circle of radius one centred at $(0,1)$



$$\text{area} = A = \iint_R 1\, dA = \int_{\theta=\pi/4}^{\theta=3\pi/4}\int_{r=\sqrt{2}}^{r=2\sin\theta} r\,dr\,d\theta$$

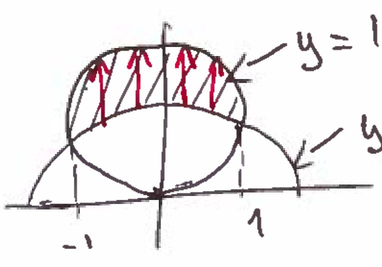
$$= \int_{\theta=\pi/4}^{\theta=3\pi/4} \left[\frac{r^2}{2}\right]_{\sqrt{2}}^{2\sin\theta} d\theta = \int_{\theta=\pi/4}^{\theta=3\pi/4} 2\sin^2\theta - 1\, d\theta$$

$$\sin^2\theta = \frac{1}{2}(1-\cos 2\theta)$$

$$\Rightarrow A = \int_{\theta=\pi/4}^{\theta=\frac{3\pi}{4}} 2\left(\frac{1}{2} - \frac{1}{2}\cos 2\theta\right) - 1\, d\theta = \int_{\theta=\pi/4}^{\theta=3\pi/4} -\cos 2\theta\, d\theta = \left[-\frac{\sin 2\theta}{2}\right]_{\pi/4}^{3\pi/4}$$

$$A = -\frac{1}{2}\left(\sin\frac{3\pi}{2} - \sin\frac{\pi}{2}\right) = -\frac{1}{2}(-1-1) = 1$$

other methods: use type-I region with cartesian coordinates:



$$A = \int_{x=-1}^{x=1}\int_{y=\sqrt{2-x^2}}^{y=1+\sqrt{1-x^2}} dy\,dx = \int_{x=-1}^{x=1} 1+\sqrt{1-x^2} - \sqrt{2-x^2}\, dx$$

$x^2+(y-1)^2=1 \;|\; (y-1)^2 = 1-x^2 \;|\; y-1 = \sqrt{1-x^2}$

$$A = \int_{x=-1}^{x=1} 1 + \sqrt{1-x^2} - \sqrt{2-x^2}\, dx = \int_{x=-1}^{x=1} 1\, dx + \int_{x=-1}^{x=1} \sqrt{1-x^2}\, dx$$

$$+ \int_{x=-1}^{x=1} -\sqrt{2-x^2}\, dx$$

$$\Rightarrow \int_{x=-1}^{x=1} \sqrt{1-x^2}\, dx \Rightarrow x = \sin u \Rightarrow dx = \cos(u)\, du$$

$$x = -1 \Rightarrow u = -\pi/2$$

$$x = +1 \Rightarrow u = +\pi/2$$

$$\Rightarrow \int_{u=-\pi/2}^{u=\pi/2} \sqrt{1-\sin^2 u}\, \cos u\, du \Rightarrow 1-\sin^2(u) = \cos^2(u)$$

$$= + \int_{u=-\pi/2}^{u=\pi/2} \cos u \cos u\, du = \int_{u=-\pi/2}^{u=\pi/2} \frac{1}{2}\left(1+\cos(2u)\right) du$$

$$= \int_{u=0}^{u=\pi/2} 1+\cos(2u)\, du = \left[u + \frac{\sin(2u)}{2}\right]_0^{\pi/2} = \frac{\pi}{2}$$

So $A = \int_{-1}^{1} 1\,dx + \frac{\pi}{2} + \int_{-1}^{1} -\sqrt{2-x^2}\,dx$

$= 2 + \frac{\pi}{2} + \int_{-1}^{1} -\sqrt{2-x^2}\,dx \Rightarrow x = \sqrt{2}\sin u$

$dx = \sqrt{2}\cos u\,du$

$= 2 + \frac{\pi}{2} + \int -\sqrt{2-2\sin^2 u}\,\sqrt{2}\cos u\,du$

$= 2 + \frac{\pi}{2} + \int -\sqrt{2(1-\sin^2 u)}\,\sqrt{2}\cos u\,du$

$= 2 + \frac{\pi}{2} + \int_{u=-\frac{\pi}{4}}^{u=\pi/4} -2\cos^2 u\,du \Rightarrow$ $x = -1 = \sqrt{2}\sin u$

$\Rightarrow \sin u = \frac{-1}{\sqrt{2}} \Rightarrow u = -\frac{\pi}{4}$

$\sin u = \frac{-\sqrt{2}}{2} \Rightarrow u = -\frac{\pi}{4}$

$= 2 + \frac{\pi}{2} + 2\int_{0}^{\pi/4} -2\cos^2 u\,du$

$x = +1 \Rightarrow u = \frac{\pi}{4}$

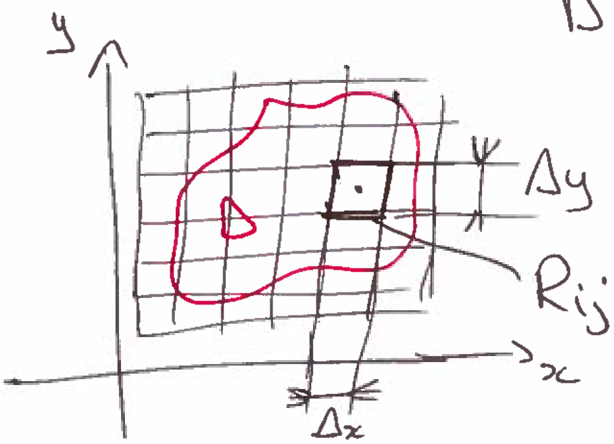
$= 2 + \frac{\pi}{2} - \frac{4}{2}\int_{0}^{\pi/4} 1 + \cos 2u\,du = 2 + \frac{\pi}{2} - 2\left[u + \frac{\sin 2u}{2}\right]_{0}^{\pi/4} du$

So $A = 2 + \frac{\pi}{2} - 2\left(\frac{\pi}{4} + \frac{\sin \pi/2}{2}\right) = 2 + \frac{\pi}{2} - \frac{\pi}{2} - 1 = \underline{1}$

Applications of double integrals

$\boxed{\text{Lecture 28}}$

Recall definition: $\iint_D f(x,y)\,dA = \lim_{\substack{m\to\infty \\ n\to\infty}} \sum_{i=1}^{n} \sum_{j=1}^{m} f(x_i^*, y_j^*)\,\Delta x \Delta y$



Suppose now D is a lamina with mass density function $\rho(x,y)$

$\Rightarrow$ mass of R_{ij} is $\approx \rho(x_i, y_j)\,\Delta x \Delta y$

$\Rightarrow$ Total mass of lamina $m = \iint_D \rho(x,y)\,dA$

$\Rightarrow$ other densities:
- charge density
- population density
- probability density

$\rho : \frac{\text{mass}}{\text{area}}$

$\rho : \frac{\text{charge}}{\text{area}}$

$\rho : \frac{\text{individuals}}{\text{area}}$

$\rho(x,y)\,dA$ = prob (x,y) is in dA