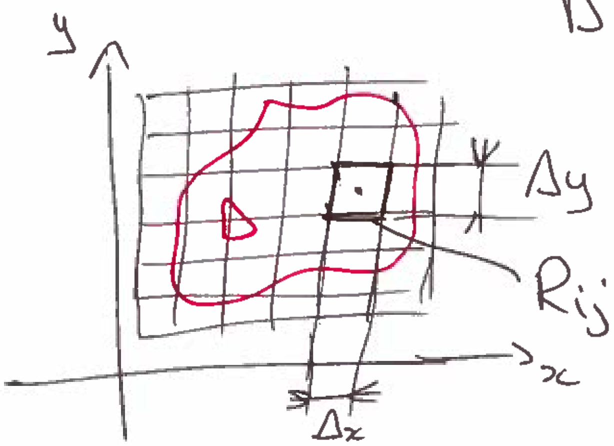
So $A = 2 + \frac{\pi}{2} - 2\left(\frac{\pi}{4} + \frac{\sin \pi/2}{2}\right) = 2 + \frac{\pi}{2} - \frac{\pi}{2} - 1 = \underline{1}$

Applications of double integrals

Lecture 28

Recall definition: $\iint_D f(x,y)\,dA = \lim_{\substack{m\to\infty \\ n\to\infty}} \sum_{i=1}^{n}\sum_{j=1}^{m} f(x_i^*, y_j^*)\Delta x \Delta y$



Suppose now D is a lamina with mass density function $\rho(x,y)$

$\Rightarrow$ mass of R_{ij} is $\approx \rho(x_i, y_j)\Delta x \Delta y$

$\Rightarrow$ Total mass of lamina $m = \iint_D \rho(x,y)\,dA$

$\Rightarrow$ other densities:
* charge density
* population density
* probability density

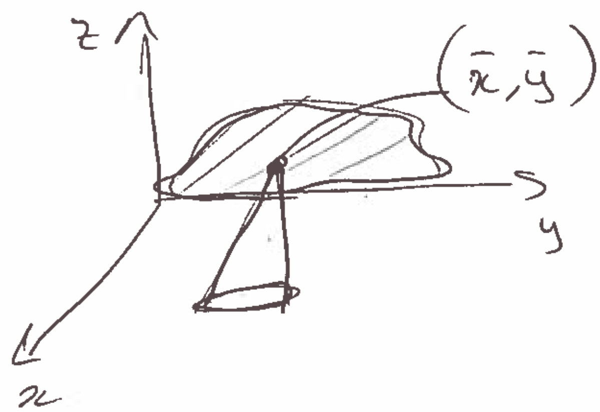
ρ: $\frac{\text{mass}}{\text{area}}$

ρ: $\frac{\text{charge}}{\text{area}}$

ρ: $\frac{\text{individuals}}{\text{area}}$

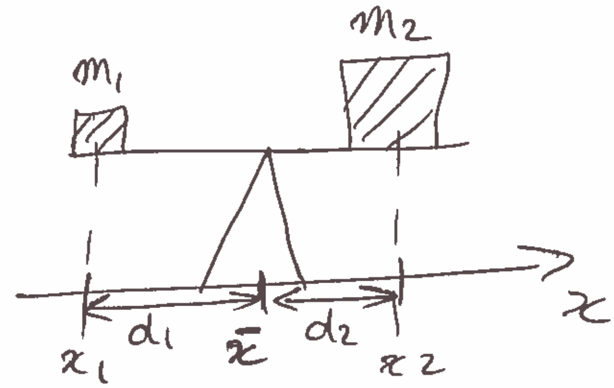
$\rho(x,y)\,dA$ = prob (x,y) is in dA

centre of mass



centre of mass = balance point

Balance on teeter-totter



balance when $m_1 d_1 = m_2 d_2$

$\Rightarrow \bar{x}$ obeys

$$m_1(\bar{x} - x_1) = m_2(x_2 - \bar{x})$$

$$\Rightarrow m_1(x_1 - \bar{x}) + m_2(x_2 - \bar{x}) = 0$$

$\Rightarrow$ in general, for more masses $\sum_{i=1}^{n} m_i(x_i - \bar{x}) = 0$

or $\sum_{i=1}^{n} m_i x_i = \bar{x} \sum_{i=1}^{n} m_i$

$$\Rightarrow \bar{x} = \frac{\sum_{i=1}^{n} m_i x_i}{\sum_{i=1}^{n} m_i}$$

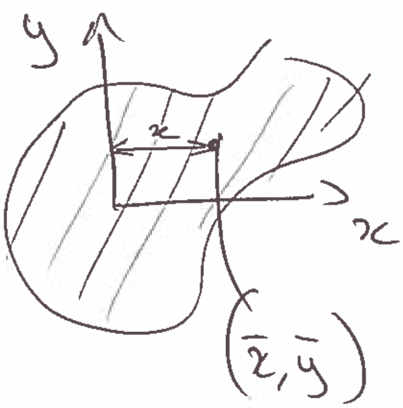
$\leftarrow$ moment

$\leftarrow$ total mass

for a continuous rod of density $\rho(x)$ $a \le x \le b$, this becomes

$$\boxed{\bar{x} = \frac{\int_a^b x\,\rho(x)\,dx}{\int_a^b \rho(x)\,dx}}$$

For a lamina D of density $\rho(x,y)$ this becomes



$$\bar{x} = \frac{\iint_D x\,\rho(x,y)\,dx\,dy}{\iint_D \rho(x,y)\,dx\,dy}$$

$\leftarrow$ moment about the y-axis

$\leftarrow$ total mass

$$\bar{y} = \frac{\iint_D y\,\rho(x,y)\,dx\,dy}{\iint_D \rho(x,y)\,dx\,dy}$$

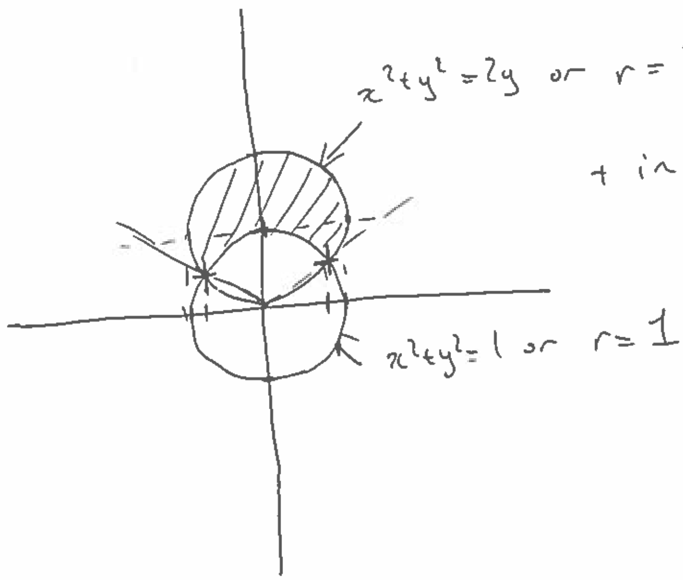
$\leftarrow$ moment about the x-axis

$\leftarrow$ total mass

Example : Find center of mass $(\bar{x},\bar{y})$ of region inside $x^2+y^2=2y$ and outside $x^2+y^2=1$, if the density is inversely proportional to distance from the origin. So $\rho(x,y) = \frac{k}{\sqrt{x^2+y^2}} \Rightarrow \frac{k}{r}$

$x^2+y^2=2y$

↳ this is a circle of radius one centered at $(0,1)$



+ intersection points : solve

$$\begin{cases} x^2+y^2=2y \\ x^2+y^2=1 \end{cases} \quad 2y=1 \Rightarrow y=\frac{1}{2}$$

$$\frac{1}{4}+x^2=1 \Rightarrow x=\pm\frac{\sqrt{3}}{2}$$

so for the circle of radius one centred at the origin we have

$$\left.\begin{aligned} x &= 1.\cos\theta = \pm\frac{\sqrt{3}}{2} \\ y &= 1.\sin\theta = \frac{1}{2} \end{aligned}\right\} \Rightarrow \theta = \frac{\pi}{6}$$

or $\theta = \frac{\pi}{6}+\frac{\pi}{6}$ ~~6~~

$\theta = \frac{5\pi}{6}$

$\Rightarrow$ now compute total mass

$$M = \text{mass} = \int_{\pi/6}^{5\pi/6}\int_{1}^{2\sin\theta} \frac{k}{r} r\,dr\,d\theta = \int_{\pi/6}^{5\pi/6} k(2\sin\theta - 1)\,d\theta = k\left[-2\cos\theta - \theta\right]_{\frac{\pi}{6}}^{5\pi/6}$$

$$M = k\left[-2\cos\left(\frac{5\pi}{6}\right) - \frac{5\pi}{6} + 2\cos\frac{\pi}{6} + \frac{\pi}{6}\right] = k\left[2\frac{\sqrt{3}}{2} + \frac{2\sqrt{3}}{2} - \frac{2\pi}{3}\right] = k\left(2\sqrt{3} - \frac{2\pi}{3}\right)$$

now $$\bar{y} = \frac{1}{M}\iint_D y\rho(x,y)\,dA = \frac{1}{M}\int_{\frac{\pi}{6}}^{5\pi/6}\int_{1}^{2\sin\theta} r\sin\theta\,\frac{k}{\cancel{r}}\,\cancel{r}\,dr\,d\theta = \frac{k}{M}\int_{\frac{\pi}{6}}^{5\pi/6}\sin\theta\int_{1}^{2\sin\theta} r\,dr\,d\theta$$

$$\bar{y} = \frac{k}{M}\int_{\pi/6}^{5\pi/6} \sin\theta \left[\frac{1}{2} r^2\right]_1^{4\sin\theta} d\theta = \frac{k}{2M}\int_{\pi/6}^{5\pi/6} \sin\theta\left(4\sin^2\theta - 1\right) d\theta \qquad \left(\sin^2\theta = 1-\cos^2\theta\right)$$

$$= \frac{k}{2M}\int_{\pi/6}^{5\pi/6} \sin\theta\left[4\left(1-\cos^2\theta\right)-1\right] d\theta = \frac{k}{2M}\int_{\pi/6}^{5\pi/6}\left(4\sin\theta - 4\sin\theta\cos^2\theta - \sin\theta\right) d\theta$$

$$= \frac{k}{2M}\int_{\pi/6}^{5\pi/6} 3\sin\theta\, d\theta - \frac{4k}{2M}\int_{\pi/6}^{5\pi/6} \sin\theta\cos^2\theta\, d\theta$$

use here

$u = \cos\theta$

$du = -\sin\theta\, d\theta$

$\theta = \pi/6 \Rightarrow u = \cos\pi/6 = \frac{\sqrt{3}}{2}$

$\theta = 5\pi/6 \Rightarrow u = \cos\frac{5\pi}{6} = -\frac{\sqrt{3}}{2}$

$$= \frac{k}{2M}\int_{\pi/6}^{5\pi/6} 3\sin\theta\, d\theta + \frac{4k}{2M}\int_{\frac{\sqrt{3}}{2}}^{-\sqrt{3}/2} u^2\, du$$

$$= -\frac{3k}{2M}\left[\cos\theta\right]_{\pi/6}^{5\pi/6} + \frac{2k}{M3}\left[u^3\right]_{\sqrt{3}/2}^{-\sqrt{3}/2}$$

$$= -\frac{3k}{2M}\left(\cos 5\pi/3 - \cos\pi/6\right) + \frac{2k}{3M}\left(-\frac{3\sqrt{3}}{8} - \frac{3\sqrt{3}}{8}\right)$$

$$= \frac{3\sqrt{3}k}{2M} - \frac{4k}{3M}\left(\frac{3\sqrt{3}}{8}\right) = \sqrt{3}\left(\frac{3}{2M} - \frac{1}{2M}\right)k \qquad M = k\left[2\sqrt{3} - \frac{2\pi}{3}\right]$$

$$\bar{y} = \frac{\sqrt{3}k}{k\left(2\sqrt{3} - 2\pi/3\right)}\left(\frac{1}{2}\right) = \frac{\sqrt{3}}{2\sqrt{3} - 2\pi/3} \simeq 1{,}26$$

and $\bar{x} = 0$ thank to the symmetry of the problem.