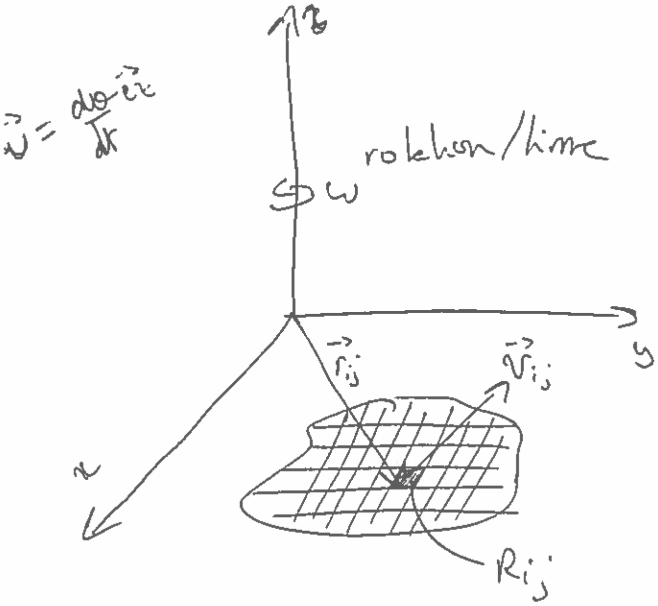
lecture 29

①

__Moment of inertia:__ Suppose a lamina is rotating about an orthogonal axis at constant rotational speed ω (radian/s) and determine it's kinetic energy K.



energy of element R_{ij} is

$$\frac{1}{2}\underbrace{\rho(x_i^*, y_j^*)\Delta x\Delta y}_{\text{mass}}|\vec{v_{ij}}|^2$$

$\vec{v_{ij}} = v_{ij\theta}\vec{e_\theta}$ (only tangential component)

$\vec{r_{ij}} = r_{ij}\vec{e_r}(\theta)$

$v_\theta = \omega r_{ij} \Rightarrow |\vec{v_{ij}}| = v_\theta^2 = \omega^2 r_{ij}^2$
$= \omega^2(x_i^2 + y_j^2)$

$\Rightarrow$ energy of element R_{ij} is

$$\frac{1}{2}\rho(x_i^*, y_j^*)\Delta x\Delta y\, \omega^2 (x_i^{*2} + y_j^{*2})$$

So $$K = \frac{1}{2}\underbrace{\iint_D (x^2+y^2)\rho(x,y)\,dA}_{= I_0 = \text{moment of inertia about the origin.}}\omega^2$$

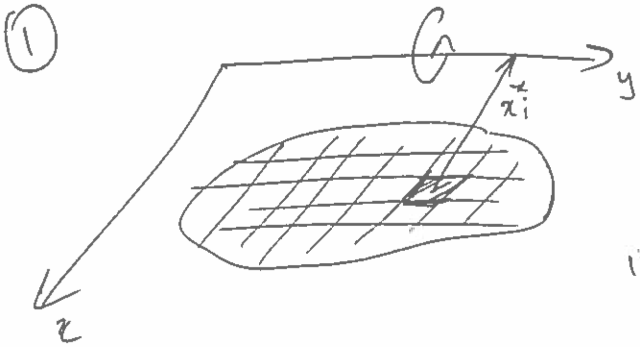
$[I_0] = kg\, m^2$

$$\boxed{K = \frac{1}{2}I_0\omega^2}$$

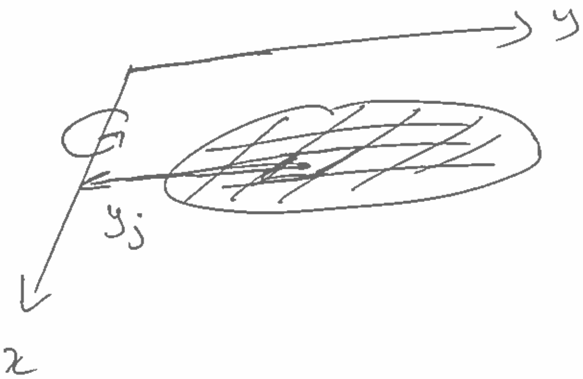
So larger $I_0 \Rightarrow$ more work (energy) needed to rotate.

$[K] = kg\, m^2 s^{-2} = J$

Two facts ①



moment of inertia about y-axis is $I_y = \iint_D x^2 \rho(x,y)\,dA$



moment of inertia about x-axis

$$I_x = \iint_D y^2 \rho(x,y)\,dA$$

so $I_0 = I_x + I_y$

② Write $\vec{r} = \vec{r}_c + \vec{r}\,'$ where $\vec{r}_c = \langle \bar{x}, \bar{y} \rangle$ is the centre of mass

and $\vec{r}\,' = \langle x - \bar{x}, y - \bar{y} \rangle$

then $|\vec{r}|^2 = |\vec{r}_c|^2 + |\vec{r}\,'|^2 + 2\underbrace{\vec{r}_c \cdot \vec{r}\,'}_{\bar{x}(x-\bar{x}) + \bar{y}(y-\bar{y})}$

so $I_0 = \iint_D |\vec{r}|^2 \rho(x,y)\,dA = M|\vec{r}_c|^2 + 0 + I_{0,c}$

because $\iint (x-\bar{x})\,\rho(x,y)\,dA = 0$

$\bar{x} = \frac{1}{M}\iint x\rho(x,y)\,dA$

$$I_0 = M|\vec{r}_c|^2 + I_{0,c}$$

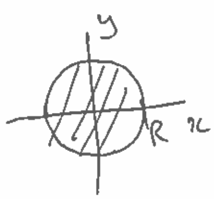
total mass

centre of mass

moment of inertia about centre of mass.

$$I_{0,c} = \iint_D \left((x-\bar{x})^2 + (y-\bar{y})^2\right) \rho(x,y)\, dA$$

Examples (1) Uniform circular disk, mass M, axis through center.



$$\rho = \frac{M}{\text{area}} = \frac{M}{\pi R^2}$$

ρ is constant here.

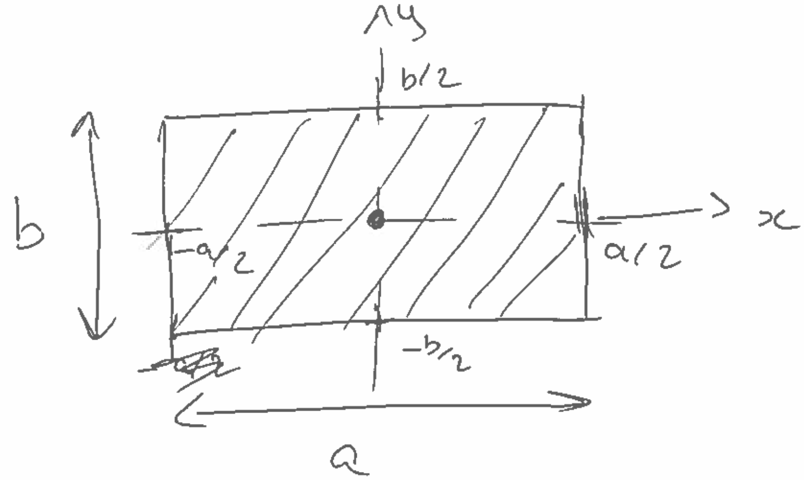
$$\Rightarrow I_0 = \iint_D (x^2+y^2)\rho\, dA = \rho \int_0^{2\pi}\int_0^R r^2\, r\, dr\, d\theta = \rho \int_0^{2\pi} \left[\frac{r^4}{4}\right]_0^R d\theta$$

$$I_0 = 2\pi\rho \frac{R^4}{4} = \frac{2\pi M R^4}{4\pi R^2} = \boxed{\frac{1}{2} M R^2 = I_0}$$

$\Rightarrow$ By symmetry, $I_x = I_y$ so $I_x = I_y = \frac{1}{2} I_0 = \frac{1}{4} M R^2$

moment of inertia for rotation about a diametral axis.

2. Uniform rectangular plate, mass M, axis through centre.



$$\rho = \frac{M}{ab}$$

$$I_y = \iint_D x^2 \rho \, dA = \frac{M}{ab} \int_{-b/2}^{b/2}\int_{-a/2}^{a/2} x^2 \, dx \, dy = \frac{M}{ab} 2 \int_{-b/2}^{b/2} \underbrace{\int_0^{a/2} x^2 \, dx}_{\left[\frac{x^3}{3}\right]_0^{a/2}} dy$$

$$I_y = \frac{2M}{3ab} \int_{-b/2}^{b/2} \frac{a^3}{8} \, dy = \frac{1 M a^3}{12 ab} \underbrace{\int_{-b/2}^{b/2} dy}_{= b}$$

$$I_y = \frac{M a^{\cancel{3}2} \cancel{b}}{\cancel{a}\cancel{b} 12} = \boxed{\frac{a^2 M}{12} = I_y}$$

Similarly $\boxed{I_x = \frac{b^2 M}{12}}$ ✓ So it is harder to rotate about the y axis than x-axis as ~~...~~ . $a > b$.

So $I_0 = I_x + I_y = \frac{1}{12} M (a^2 + b^2)$

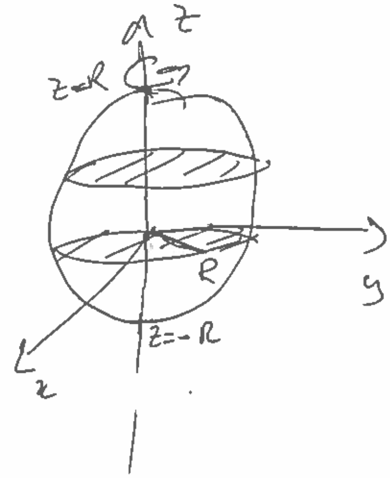
$\Rightarrow$ (same as for a disk of radius $\sqrt{\frac{a^2+b^2}{6}}$

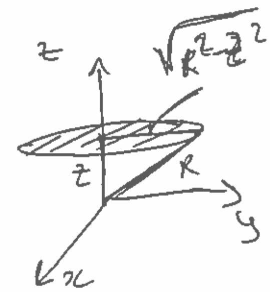
or diameter $\sqrt{\frac{2}{3}(a^2+b^2)}$)

3 Solid uniform sphere

Exercise: show $I = \frac{2}{5} M R^2$

Hint: integrate over vertical slices.





$$I(z) = \frac{1}{2} M(z) R(z)^2 = \frac{1}{2} \rho \pi R(z)^2 R(z)^2 = \frac{1}{2} \rho \pi \left(R^2 - z^2\right)^2$$

$$I = \int_{-R}^{R} I(z)\, dz = \int_{-R}^{R} \frac{1}{2} \rho \pi \left(R^2 - z^2\right)^2 = \cancel{2} \int_{0}^{R} \frac{1}{\cancel{2}} \rho \pi \left(R^4 - 2R^2 z^2 + z^4\right) dz$$

$$I = \rho \pi \left[\left[z R^4 \right]_0^R - 2R^2 \left[\frac{z^3}{3} \right]_0^R + \left[\frac{z^5}{5} \right]_0^R \right]$$

$$= \rho \pi \left(R^5 - \frac{2}{3} R^5 + \frac{R^5}{5} \right) = \rho \pi \frac{8}{15} R^2 = \rho \left(\frac{4}{3} \pi R^3 \right) \left(\frac{2}{5} R^2 \right) = \boxed{\frac{2 M R^2}{5} = I}$$