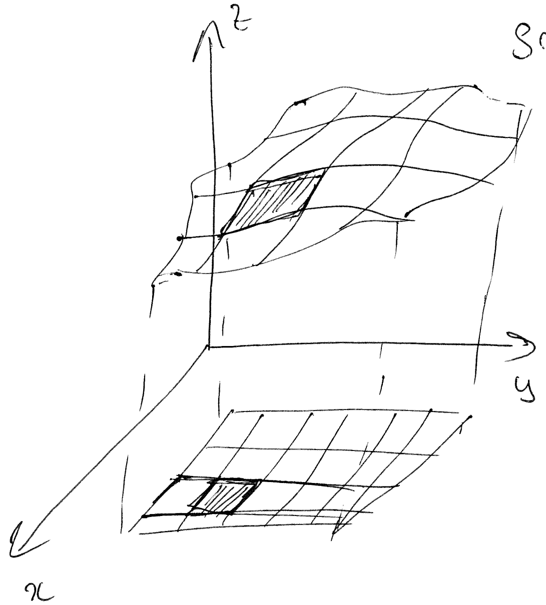
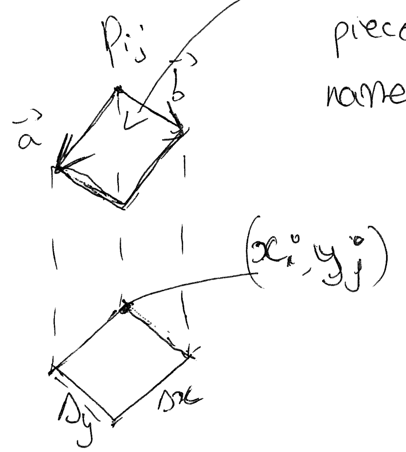
Lecture 30 ①

<u>Surface Area</u>: Goal: determine area of surface $z = f(x,y)$ = sum of areas above small rectangles.



area above of small rectangle $\approx$ area of local tangent plane.



approximate little piece of surface by little piece of tangent plane at P_{ij} above (x_i, y_j), namely a little parallelogram of area ΔT_{ij}

In ~~xz~~ plane, tangent line in direction $\vec{a}$ has slope $f_x(x_i, y_j)$ so

$$\vec{a} = \Delta x\left(\vec{i} + f_x(x_i, y_j)\vec{k}\right) = \Delta x \langle 1, 0, f_x \rangle$$

similarly $\vec{b} = \Delta y\left(\vec{j} + f_y(x_i, y_j)\vec{k}\right) = \Delta y \langle 0, 1, f_y \rangle$

Then $\Delta T_{ij} = |\vec{a} \times \vec{b}| = \Delta x \Delta y \,|\langle 1, 0, f_x \rangle \times \langle 0, 1, f_y \rangle|$

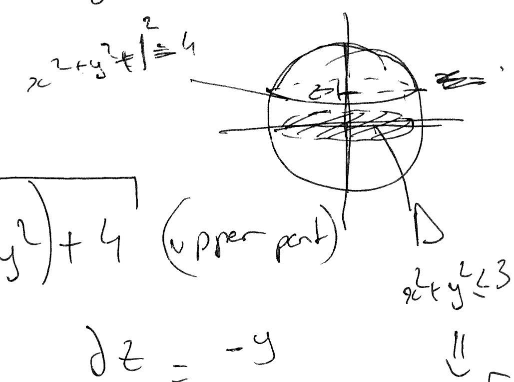
and since $\langle 1, 0, f_x \rangle \times \langle 0, 1, f_y \rangle = \begin{vmatrix} i & j & k \\ 1 & 0 & f_x \\ 0 & 1 & f_y \end{vmatrix} = \langle -f_x, -f_y, 1 \rangle$

we get: $\Delta T_{ij} = \Delta x \Delta y \sqrt{1 + f_x(x_i, y_j)^2 + f_y(x_i, y_j)^2}$

and total surface area $\approx \sum_i^m \sum_j^n \Delta T_{ij} = \sum_i^m \sum_j^n \sqrt{1 + f_x(x_i, y_j)^2 + f_y(x_i, y_j)^2}\, \Delta x \Delta y$

Area $A = \iint_D \sqrt{1 + f_x(x,y)^2 + f_y(x,y)^2}\, dA = \iint_D \sqrt{1 + \left(\frac{\partial z}{\partial x}(x,y)\right)^2 + \left(\frac{\partial z}{\partial y}(x,y)\right)^2}\, dA$

Example: Find area of part of sphere $x^2 + y^2 + z^2 = 4$ above $z = 1$



$A = \iint_D \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\, dA$, here $z = \sqrt{-(x^2+y^2)+4}$ (upper part)

so $\frac{\partial z}{\partial x} = \frac{-x}{\sqrt{4-x^2-y^2}}$ $\quad \frac{\partial z}{\partial y} = \frac{-y}{\sqrt{4-x^2-y^2}}$

So $A = \iint_D \sqrt{1 + \frac{x^2}{4-x^2-y^2} + \frac{y^2}{4-x^2-y^2}}\, dA = \iint \sqrt{\frac{4 \cancel{-x^2} \cancel{-y^2} \cancel{+x^2} \cancel{+y^2}}{4-x^2-y^2}}\, dA$

$A = \iint \sqrt{\frac{4}{4-x^2-y}}\, dA = \underbrace{\int_0^{2\pi} \int_0^{\sqrt{3}} \frac{2}{\sqrt{4-r^2}}\, r\,dr\,d\theta}_{\text{use}}$ $\quad u = r^2 \Rightarrow du = 2r\,dr$

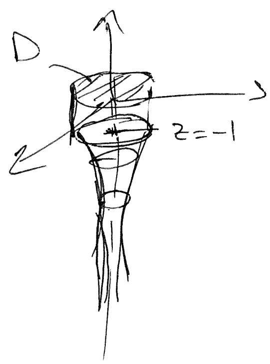
$A = \int_0^{2\pi} \int_0^{3} \frac{1}{\sqrt{4-u}}\, du\, d\theta$

$$A = \int_0^{2\pi} \Big[-2(4-u)^{1/2} \Big]_0^3 \, d\theta = \int_0^{2\pi} -2(1-2) \, d\theta$$

$$\boxed{A = 4\pi}$$

<u>Example</u>: Consider the funnel shaped region bounded by

$z = \dfrac{-1}{\sqrt{x^2+y^2}}$ and $z=-1$ $\quad R = \{(x,y) \;\; x^2+y^2 \leq 1\}$



infinitely long paint can.

$\Rightarrow$ find it's volume and it's surface area.

Volume $V = -\iint \dfrac{-1}{\sqrt{x^2+y^2}} \, dA - \pi$ ← Volume of cylinder above $z=-1$ $(\pi R^2 . H = \pi$, $R=1, H=1)$

$$= \int_0^{2\pi}\int_0^1 \frac{1}{r} \, r \, dr \, d\theta - \pi = 2\pi - \pi = \pi \, (cm^3)$$

Now compute its' surface area: