Lecture 31

Now compute it's surface area $\quad z = \frac{-1}{\sqrt{x^2+y^2}}$

$$\frac{\partial z}{\partial x} = \frac{x}{(x^2+y^2)^{3/2}} \qquad \frac{\partial z}{\partial y} = \frac{y}{(x^2+y^2)^{3/2}}$$

$$\Rightarrow \quad 1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2 = 1+\frac{x^2}{(x^2+y^2)^3}+\frac{y^2}{(x^2+y^2)^3} = 1+\frac{1}{(x^2+y^2)^2} = 1+\frac{1}{r^4}$$

So Area $A = \iint_D \sqrt{1+\frac{1}{r^4}}\, dA$

$$= \int_0^{2\pi}\int_0^1 \sqrt{1+\frac{1}{r^4}}\, r\,dr\,d\theta = \int_0^{2\pi}\int_0^1 \frac{1}{r^2}\sqrt{r^4+1}\, r\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^1 \frac{1}{r}\sqrt{r^4+1}\,dr\,d\theta = 2\pi \underbrace{\int_0^1 \frac{1}{r}\sqrt{r^4+1}\,dr}_{\text{integral diverges to } +\infty}$$

$$\int_{r=0}^{r=1} \frac{1}{r}\sqrt{r^4+1}\,dr \geq \int_{r=0}^{r=1} \frac{1}{r}\,dr = \Big[\ln r\Big]_0^1 = 0 - \lim_{\varepsilon\to 0} \ln\varepsilon = \infty$$

so $A = \infty$

but $V = \pi < \infty$ (!!)

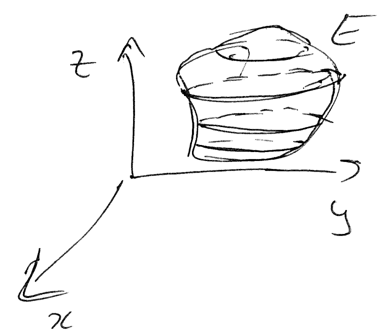
So  is finite volume π but indefinite surface area. It is a paint can that we can fill with paint but we can't paint it. What is going on? Answer: A finite volume of paint can paint an infinite surface area ~~ff~~ as long as we can spread the paint arbitrarily thin (just keep spreading it out!). Of course real paint has a minimal thickness, but then such paint would get stuck in the neck of the can when the neck narrows to less than the thickness.

Triple integrals

function of 3 variables

Goal: define integral $\displaystyle\iiint_E f(x,y,z)\,dV$

region in $\mathbb{R}^3$

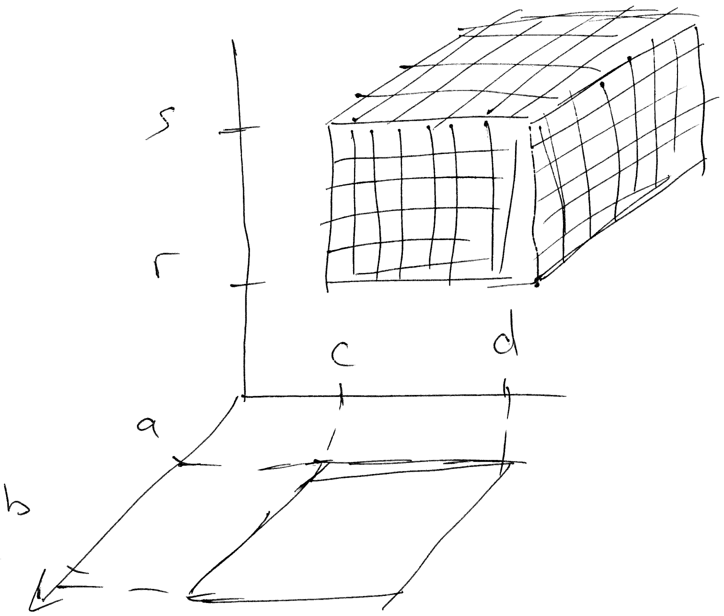


example $f(x,y,z)$ = mass density

$\to$ then $\iiint f(x,y,z)\,dV$ = total mass.

or $f(x,y,z) = 1$, then $\displaystyle\iiint_E dV$ = Volume of E

Simplest region: Box $B = [a,b] \times [c,d] \times [r,s]$



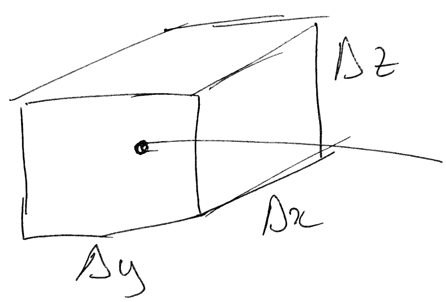
~~divide [a,b] in ℓ subintervals~~

divide $[a,b]$ into ℓ sub-intervals of length $\Delta x = \dfrac{b-a}{\ell}$

" $[c,d]$ " m " " " $\Delta y = \dfrac{d-c}{m}$

" $[r,s]$ " n " " " $\Delta z = \dfrac{s-r}{n}$

$\Rightarrow$ typical subbox B_{ijk}



some point $(x_{ijk}^*, y_{ijk}^*, z_{ijk}^*)$

eg corner (x_i, y_j, z_k)

Then define $$\iiint_B f(x,y,z)\,dV = \lim_{\ell,m,n\to\infty} \sum_i^{\ell} \sum_j^{m} \sum_k^{n} f\underbrace{(x_{ijk}^*, y_{ijk}^*, z_{ijk}^*)}_{\text{a point in the subrectangle}} \underset{\Delta x \Delta y \Delta z}{\Delta V}$$

limit exists if f is nice, e.g continuous and is indep of choice of $(x_{ijk}^*, y_{ijk}^*, z_{ijk}^*)$ (4)

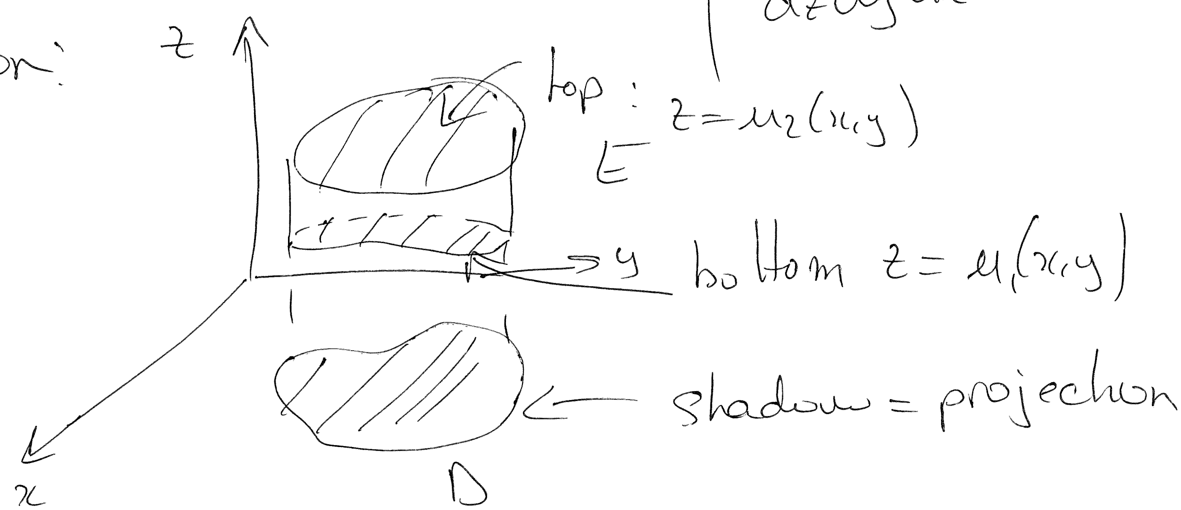
Fubini's theorem: $\iiint_B f(x,y,z)\,dV = \int_r^s\int_c^d\int_a^b f(x,y,z)\,dx\,dy\,dz$

= 5 other iterated integrals.

$dx\,dz\,dy$
$dy\,dx\,dz$
$dy\,dz\,dx$
$dz\,dx\,dy$
$dz\,dy\,dx$

more general regions:

Type 1 region:



$E = \{(x,y,z) : (x,y)\in D,\ u_1(x,y) \le z \le u_2(x,y)\}$

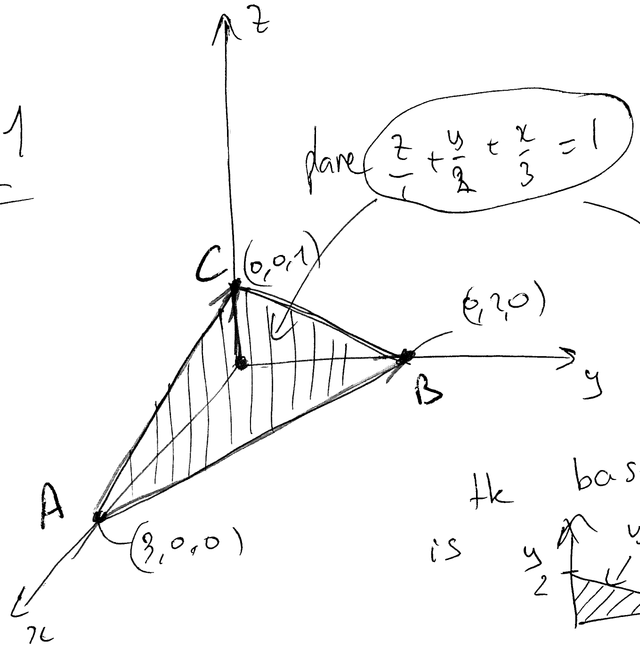
(D is typically Type I or Type II in xy plane)

Type 2 : $E = \{(x,y,z) : (y,z) \in D : u_1(y,z) \leq x \leq u_2(y,z)\}$

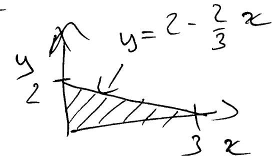
Type 3 : $E = \{(x,y,z) : (x,z) \in D : u_1(x,z) \leq y \leq u_2(x,z)\}$.

Example : Evaluate $I = \iiint_E e^{x+y+z}\, dV$ E is the tetrahedron with vertices $(3,0,0), (0,2,0), (0,0,1), (0,0,0)$

Type 1



the base is



Type I

→ to find the plane

$\vec{AB} = (-3, 2, 0)$

$\vec{AC} = (-3, 0, 1)$

$\vec{n}$ at A is

$\vec{AB} \times \vec{AC} = 2\vec{i} + 3\vec{j} + 6\vec{k}$

remember equation of plane is $ax + by + cz = ax_0 + by_0 + cz_0$ with a, b, c the component of $\vec{n}$ $(x_0 = 3, y_0 = 0, z_0 =$

so $2x + 3y + 6z = 2 \times 3$

$\Rightarrow \boxed{\frac{x}{3} + \frac{y}{2} + z = 1}$

or $z = 1 - \frac{x}{3} - \frac{y}{2}$

$$I = \int_0^3 \int_0^{2-\frac{2}{3}x} \int_0^{1-\frac{x}{3}-\frac{y}{2}} e^x e^y e^z \, dz\, dy\, dx$$

$$= \int_0^3 \int_0^{2-\frac{2}{3}x} e^x e^y \left(e^{1-\frac{x}{3}-\frac{y}{2}} - 1\right) dy\, dx$$

$$= \int_0^3 e^x \int_0^{2-\frac{2}{3}x} \left(e^{1-\frac{x}{3}+\frac{y}{2}} - e^y\right) dy\, dx = \int_0^3 e^x \left[2e^{1-\frac{x}{3}+\frac{y}{2}} - e^y\right]_0^{2-\frac{2}{3}x} dx$$

$$= \int_0^3 e^x \left[\left(2e^{1-\frac{x}{3}+\left(2-\frac{2}{3}x\right)\frac{1}{2}} - e^{2-\frac{2}{3}x}\right) - \left(2e^{1-\frac{x}{3}} - 1\right)\right] dx$$

$$I = \int_0^3 e^x \left(2e^{1-\frac{x}{3}+1-\frac{1}{3}x} - e^{2-\frac{2}{3}x} - 2e^{1-\frac{x}{3}} + 1 \right) dx$$

$$= \int_0^3 \left(2e^{2+\frac{1}{3}x} - e^{2+1/3\,x} - 2e^{1+\frac{2x}{3}} + e^x \right) dx$$

$$= \int_0^3 \left(e^{2+1/3\,x} - 2e^{1+\frac{2}{3}x} + e^x \right) dx = \left[3e^{2+1/3\,x} - 2\left(\frac{3}{2}\right) e^{1+\frac{2}{3}x} + e^x \right]_0^3$$

$$= \cancel{3e^3} - \cancel{3e^3} + e^3 - \left(3e^2 - 3e^1 + 1 \right)$$

$$I = e^3 - 3e^2 + 3e + 1 = (e-1)^3$$