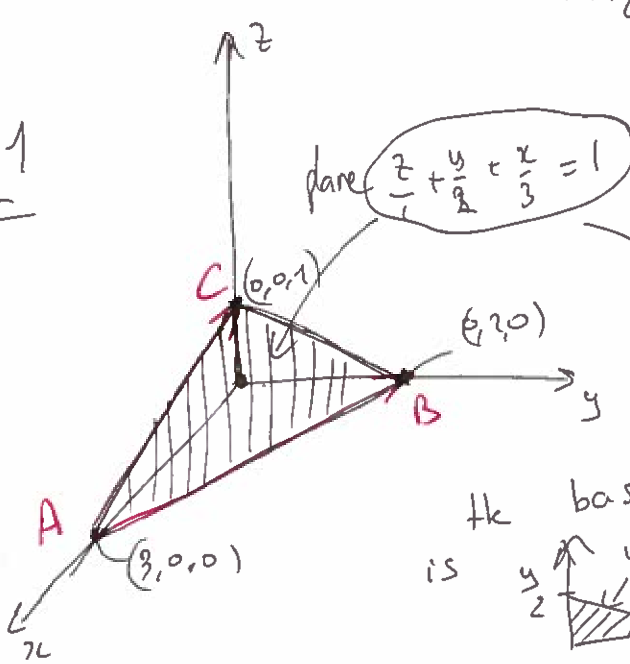
Type 2 : $E = \{ (x,y,z) : (y,z) \in D : u_1(y,z) \leq x \leq u_2(y,z) \}$ (5)

Type 3: $E = \{ (x,y,z) : (x,z) \in D : u_1(x,z) \leq y \leq u_2(x,z) \}$.

Example: Evaluate $I = \iiint_E e^{x+y+z}\, dV$ E is the tetrahedron with vertices $(3,0,0), (0,2,0), (0,0,1), (0,0,0)$

Type 1




→ to find the plane

$\vec{AB} = (-3,2,0)$

$\vec{AC} = (-3,0,1)$

$\vec{n}$ at A is

$\vec{AB} \times \vec{AC} = 2\vec{i} + 3\vec{j} + 6\vec{k}$

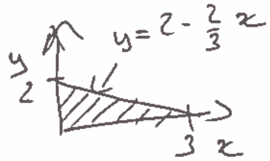
remember eq-tion of plane is $ax + by + cz = ax_0 + by_0 + cz_0$ with $a, b, c$ the component of $\vec{n}$ $(x_0 = 3, y_0 = 0, z_0 =$

so $2x + 3y + 6z = 2 \times 3$

$\Rightarrow \boxed{\frac{x}{3} + \frac{y}{2} + z = 1}$

or $z = 1 - \frac{x}{3} - \frac{y}{2}$

the base is




Type I

$$I = \int_0^3 \int_0^{2-\frac{2}{3}x} \int_0^{1-\frac{x}{3}-\frac{y}{2}} e^x e^y e^z \, dz\, dy\, dx$$

$$= \int_0^3 e^x \int_0^{2-\frac{2}{3}x} e^y \left( e^{1-\frac{x}{3}-\frac{y}{2}} - 1 \right) dy\, dx$$

$$= \int_0^3 e^x \int_0^{2-\frac{2}{3}x} \left( e^{1-\frac{x}{3}+\frac{y}{2}} - e^y \right) dy\, dx = \int_0^3 e^x \left[ 2e^{1-\frac{x}{3}+\frac{y}{2}} - e^y \right]_0^{2-\frac{2}{3}x} dx$$

$$= \int_0^3 e^x \left[ \left( 2e^{1-\frac{x}{3}+\left(2-\frac{2}{3}x\right)\frac{1}{2}} - e^{2-\frac{2}{3}x} \right) - \left( 2e^{1-\frac{x}{3}} - 1 \right) \right] dx$$

6

$$I = \int_0^3 e^{x}\left(2e^{1-\frac{x}{3}+1-\frac{1}{3}x} - e^{2-\frac{2}{3}x} - 2e^{1-\frac{x}{3}} + 1\right)dx$$

$$= \int_0^3 \left(2e^{2+\frac{1}{3}x} - e^{2+1/3\,x} - 2e^{1+\frac{2x}{3}} + e^{x}\right)dx$$

$$= \int_0^3 \left(e^{2+1/3\,x} - 2e^{1+\frac{2}{3}x} + e^{x}\right)dx = \left[3e^{2+1/3\,x} - 2\left(\frac{3}{2}\right)e^{1+\frac{2}{3}x} + e^{x}\right]_0^3$$

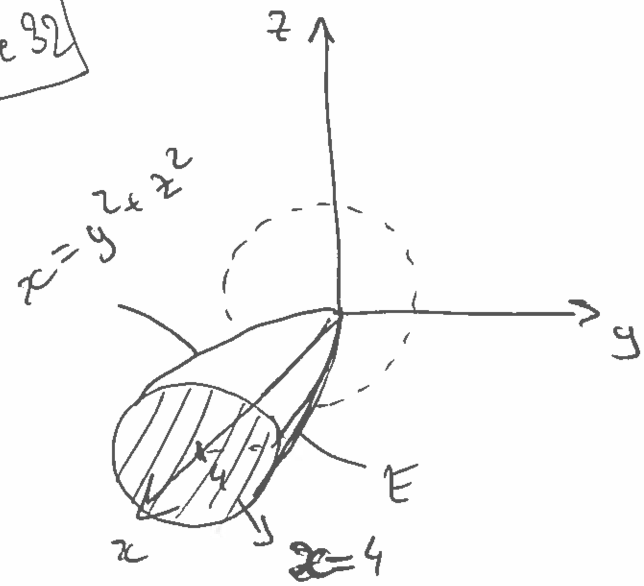
$$= \cancel{3e^3} - \cancel{3e^3} + e^3 - \left(3e^2 - 3e^1 + 1\right)$$

$$I = e^3 - 3e^2 + 3e + 1 = (e-1)^3$$

Example: Let $E$ be the solid region between $x=4$ and $x=y^2+z^2$, set up limits of integration for $I=\iiint_E f\,dV$ (1)

Sol'n Consider $E$ as Type 2 region

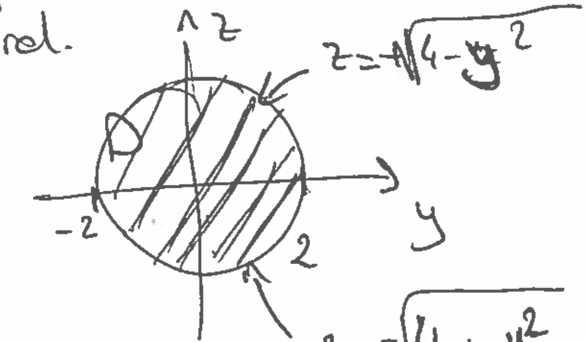
lecture 32




$$J=\iint_D \int_{x=y^2+z^2}^{x=4} f(x,y,z)\,dx\,dA$$

$(dy\,dz = dz\,dy)$

more than one way to handle this double integral.

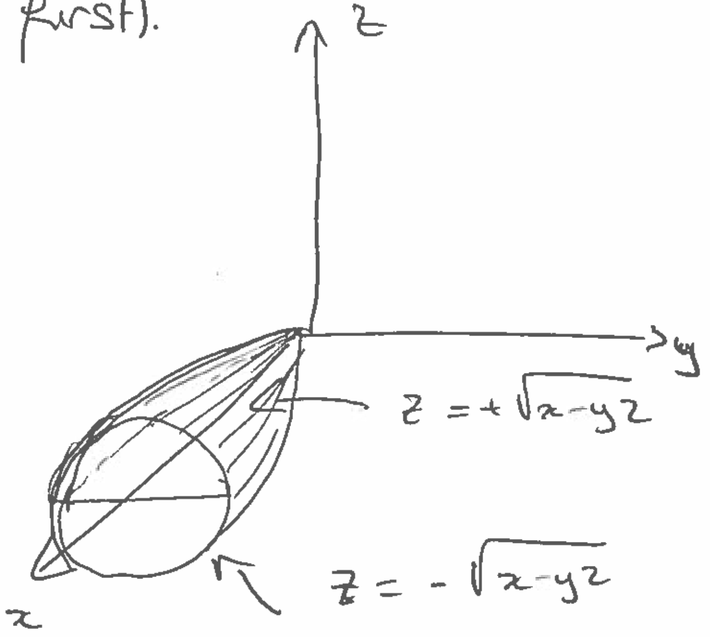



$y^2+z^2=4$

$$\text{So } J=\int_{y=-2}^{y=2}\int_{z=-\sqrt{4-y^2}}^{z=\sqrt{4-y^2}}\int_{x=y^2+z^2}^{x=4} f(x,y,z)\,dx\,dz\,dy$$

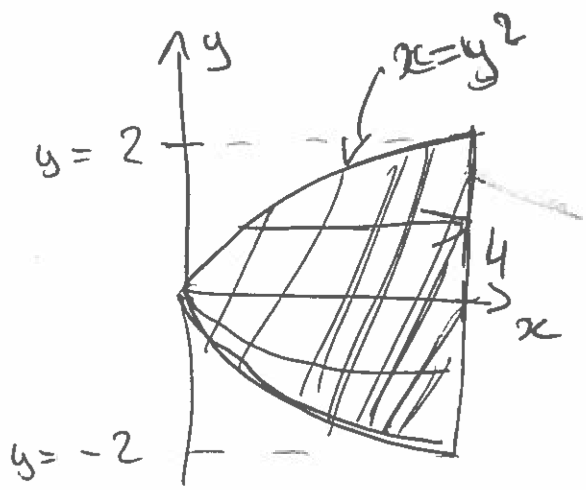
Maybe polar coordinates would be better for $\iint_D (\ )\,dA$, depending on $f(x,y,z)$ e.g if $f(x,y,z)=\sqrt{x^2+y^2+z^2}$

Sol 2: Treat $E$ as a Type 1 region (i.e, integrate in $z$-direction first).




$$I = \iint_D \int_{z=-\sqrt{x-y^2}}^{z=\sqrt{x-y^2}} f(x,y,z)\,dz\,dA$$

then for $D$:




Consider $D$ as type I or II

we consider $D$ here as type II (i.e integrate in $x$-direction first).

$$I = \int_{y=-2}^{y=2} \int_{x=y^2}^{x=4} \int_{z=-\sqrt{x-y^2}}^{z=\sqrt{x-y^2}} f(x,y,z)\,dz\,dx\,dy$$ when type II

$$I = \int_{x=0}^{x=4} \int_{y=-\sqrt{x}}^{y=\sqrt{x}} \int_{z=-\sqrt{x-y^2}}^{z=\sqrt{x-y^2}} f(x,y,z)\,dz\,dy\,dx$$ when Type I.

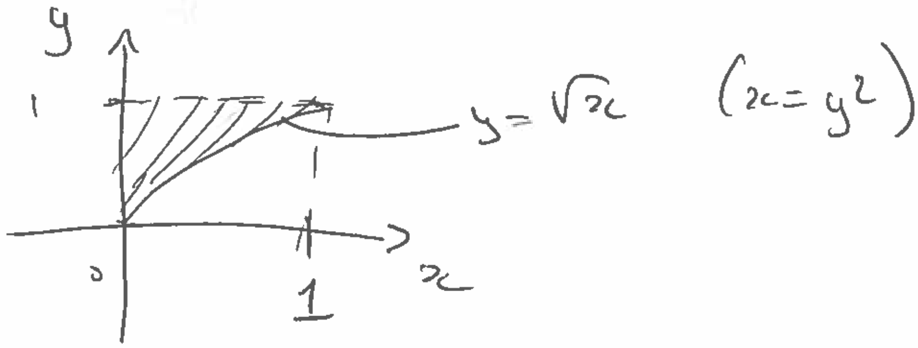
3

**Example**: let $I = \int_{x=0}^{x=1} \int_{y=\sqrt{x}}^{y=1} \int_{z=0}^{1-y} f \, dz \, dy \, dx$ ①

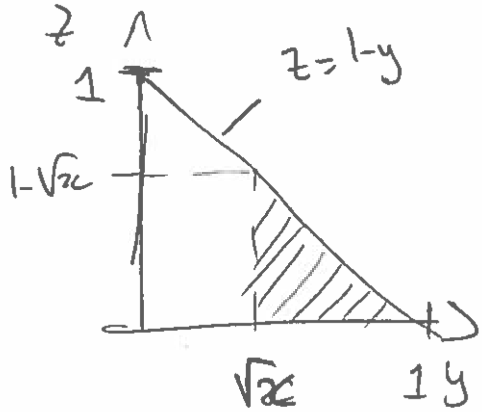
Type 1

=> write the other 5 interated integrals.

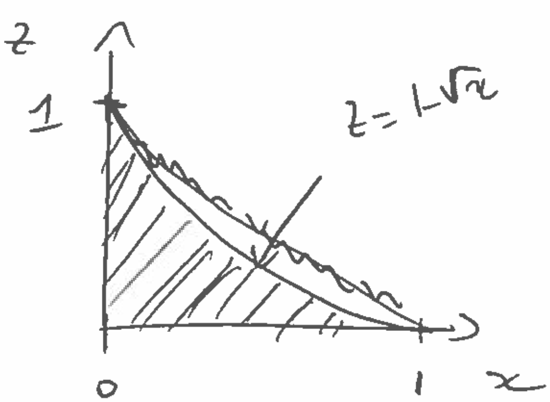
=> xy-plane




=> yz-plane




=> xz-plane




=> $z^2 = (1-\sqrt{x})^2$

=> $x = (1-z)^2$

4

$dz\,dy\,dx$ ✓ 1
$dz\,dx\,dy$ ✓ 2
$dy\,dz\,dx$ ✓ 5
$dy\,dx\,dz$ ✓ 6
$dx\,dz\,dy$ ✓ 4
$dx\,dy\,dz$ ✓ 3

$$I = \int_0^1 \int_0^{y^2} \int_{z=0}^{1-y} f\,dz\,dx\,dy \quad 2 \quad \text{type 1}$$

$$I = \int_{z=0}^{1} \int_{y=0}^{y=1-z} \int_{x=0}^{x=y^2} f\,dx\,dy\,dz \quad 3 \quad \text{type 2}$$

$D \in \mathbb{R}^2$ in $z,y$-plane

$$I = \int_{y=0}^{1} \int_{z=0}^{z=1-y} \int_{x=0}^{y^2} f\,dx\,dz\,dy \quad 4 \quad \text{type 2}$$

$D \in \mathbb{R}^2$ in $z$-$y$ plane

$$I = \int_{x=0}^{x=1} \int_{z=0}^{z=1-\sqrt{x}} \int_{y=\sqrt{x}}^{y=1-z} f\,dy\,dz\,dx \quad 5 \quad \text{type 3}$$

$D \in \mathbb{R}^2$ in $xz$ plane

$$I = \int_{z=0}^{1} \int_{0}^{(1-z)^2} \int_{\sqrt{x}}^{1-z} f\,dy\,dx\,dz \quad 6 \quad \text{type 3}$$

$D \in \mathbb{R}^2$ in $xz$ plane