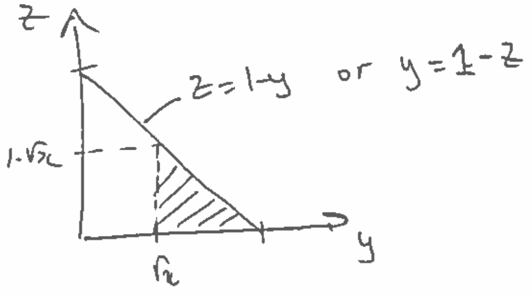
lecture 33

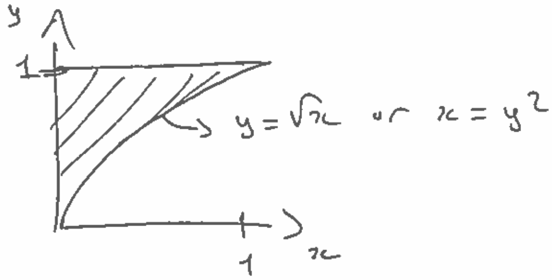
Let $I = \int_{x=0}^{x=1} \int_{y=\sqrt{x}}^{y=1} \int_{z=0}^{z=1-y} f \, dz \, dy \, dx$ ①

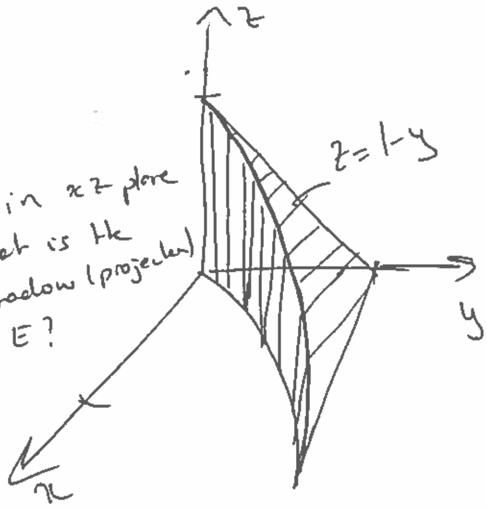
write the 5 other iterated integrals.

yz-plane



xy-plane





in xz plane what is the shadow (projection) of E?

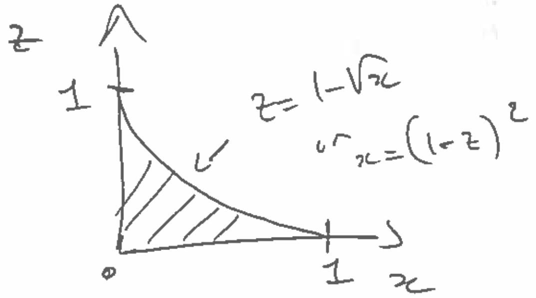
② $I = \underbrace{\int_{y=0}^{y=1} \int_{x=0}^{x=y^2}}_{\text{swapped from ①}} \int_{z=0}^{1-y} f \, dz \, dx \, dy$

$\Rightarrow$ ③ $I = \int_{y=0}^{y=1} \underbrace{\int_{z=0}^{z=1-y} \int_{x=0}^{x=y^2}}_{\text{swapped in ②}} f \, dx \, dz \, dy$ (because all upper limits depend on y only)

④ $I = \int_{z=0}^{z=1} \int^{1-z} \int_{x=0}^{x=y^2} f \, dx \, \underbrace{dy \, dz}_{\to \text{swapped in ③}}$

⑤ $$I = \int_{z=0}^{z=1}\int_{x=0}^{x=(1-z)}\int_{\sqrt{x}}^{1-z} f\,dy\,dx\,dz$$

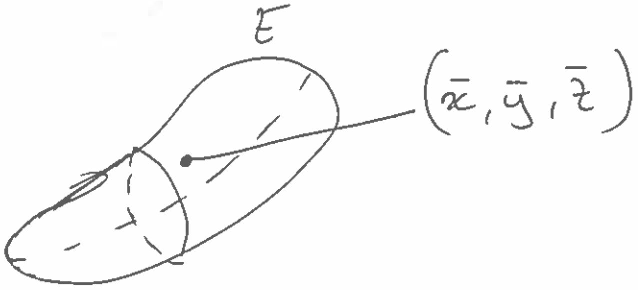
in xz plane!
the red arc lies
on $z=1-y$ and $y=\sqrt{x}$
so $z = 1-\sqrt{x}$



⑥ $$I = \int_{x=0}^{x=1}\underbrace{\int_{z=0}^{1-\sqrt{x}}\int_{\sqrt{x}}^{1-z}}_{\text{swapped from ①}} f\,dy\,dz\,dx$$

~~Centre of mass~~:

Applications: much the same as for double integrals, mass density (other density charge density). Center of mass



mass density $\rho(x,y,z)\left(\frac{kg}{m^3}\right)$

Total mass $M = \iiint_E \rho(x,y,z)\,dV$

centre of mass! $$\bar{x} = \frac{\iiint_E x\rho(x,y,z)\,dV}{\iiint_E \rho(x,y,z)\,dV}$$

← moment about yz plane

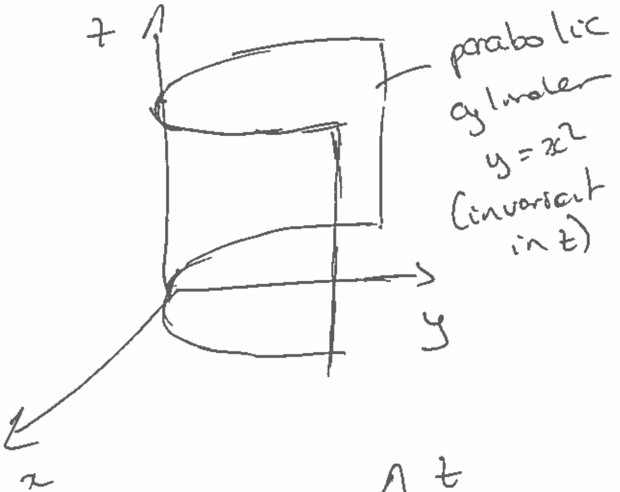
← total mass.

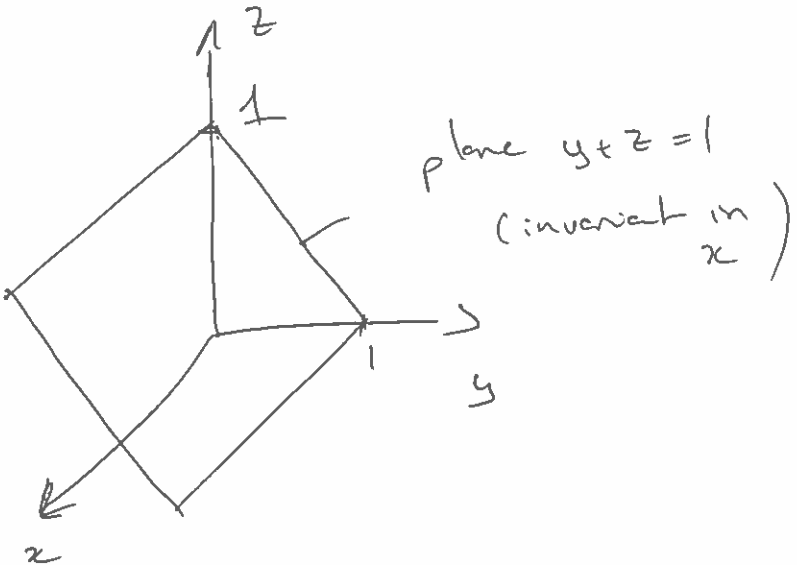
$$\bar{y} = \frac{\iiint y\rho(x,y,z)\,dV}{\iiint \rho(x,y,z)\,dV} \quad \text{and} \quad \bar{z} = \frac{\iiint z\rho(x,y,z)\,dV}{\iiint_E \rho(x,y,z)\,dV}$$

=> Special case of uniform density (constant) ρ_0 :

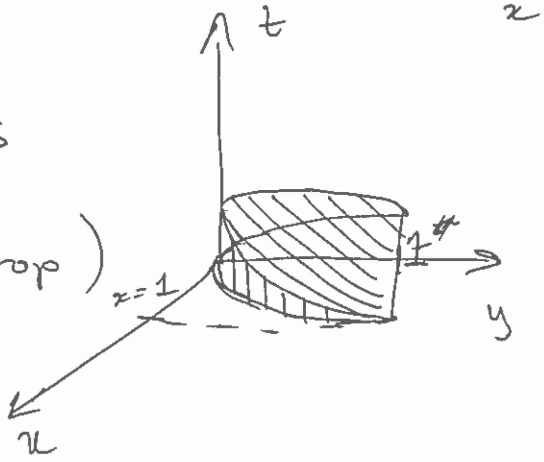
$$\bar{x} = \frac{\iiint_E x\cancel{\rho_0}\,dV}{\iiint \cancel{\rho_0}\,dV} = \frac{1}{\text{Volume}(E)} \iiint x\,dV$$ and similar for $\bar{y}, \bar{z}$.

<u>Example</u> : Find volume and centre of mass of region bounded by the parabolic cylinder $y = x^2$, the xy-plane and the plane $y+z = 1$

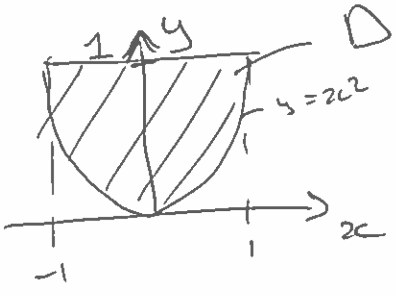




region is (door stop)



base of region



$$\text{Volume} = \int\int_D\int_{z=0}^{z=1-y} dz\,dx\,dy = \int_{-1}^{1}\int_{x^2}^{1}\int_{z=0}^{1-y} dz\,dy\,dx$$

$$\text{Volume} = \dots = \frac{8}{15}$$

$$\bar{x} = \frac{15}{8}\iiint x\,dV = 0 \quad \text{by symmetry.}$$

$$\bar{y} = \frac{15}{8}\iiint y\,dV = \int_{-1}^{1}\int_{x^2}^{1}\int_{z=0}^{1-y} y\;dz\,dy\,dx \times \frac{15}{8}$$

$$\bar{y} = \frac{15}{8}\int_{-1}^{1}\int_{x^2}^{1} y(1-y)\,dy\,dx = \frac{15}{8}\int_{-1}^{1}\int_{x^2}^{1}(y-y^2)\,dy\,dx$$

$$\bar{y} = \frac{15}{8}\int_{-1}^{1}\left[\frac{y^2}{2}-\frac{y^3}{3}\right]_{x^2}^{1} dx = \frac{15}{8}\int_{-1}^{1}\left(\frac{1}{2}-\frac{1}{3}\right)-\left(\frac{x^4}{2}-\frac{x^6}{3}\right)dx$$

$$\bar{y} = \frac{15}{8}\left[\frac{x}{2}-\frac{x}{3}-\frac{x^5}{10}+\frac{x^7}{21}\right]_{-1}^{1} = \left[\left(\frac{1}{2}-\frac{1}{3}-\frac{1}{10}+\frac{1}{21}\right)-\left(\frac{-1}{2}+\frac{1}{3}+\frac{1}{10}-\frac{1}{21}\right)\right]$$

$$= \left(1-\frac{2}{3}-\frac{2}{10}+\frac{2}{21}\right)\frac{15}{8}\,\frac{\cancel{8}}{35}\,\frac{15}{\cancel{8}} = \frac{15}{35} = \frac{3}{7}$$

$$\bar{z} = \frac{15}{8} \cancel{*} \iiint_E z\,dV = \frac{15}{8}\iint_D \underbrace{\boxed{\int_{z=0}^{1-y} z\,dz}}_{= \frac{1}{2}(1-y)^2} dA = \frac{15}{8}\int_{-1}^{1}\int_{x^2}^{1} \frac{1}{2}(1-y)^2\,dy\,dx$$

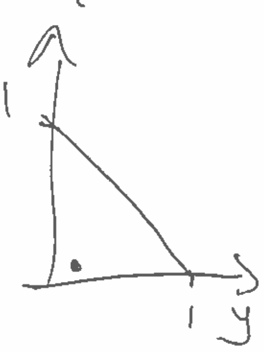
$$= \frac{15}{8}\frac{1}{2}\int_{-1}^{1} -\left[\frac{1}{3}(1-y)^3\right]_{x^2}^{1} dx = -\frac{15}{8}\frac{1}{2}\int_{-1}^{1} -\frac{1}{3}(1-x^2)^3\,dx$$

$$= +\frac{15}{48}\int_{-1}^{1}(1-x^2)^3\,dx = \frac{15}{48} * 2\int_{0}^{1} 1-3x^2+3x^4-x^6\,dx$$

$$= \frac{5}{8}\left[x - \frac{3x^3}{3} + \frac{3x^5}{5} - \frac{x^7}{7}\right]_0^1$$

$$= \frac{5}{8}\left(\cancel{1}-\cancel{1}+\frac{3}{5}-\frac{1}{7}\right) = \frac{3}{8} - \frac{5}{7\times 8} = \frac{21-5}{7\times 8} = \frac{2}{7}$$

So centre of gravity is $(\bar{x},\bar{y},\bar{z}) = \left(0, \frac{3}{7}, \frac{2}{7}\right)$

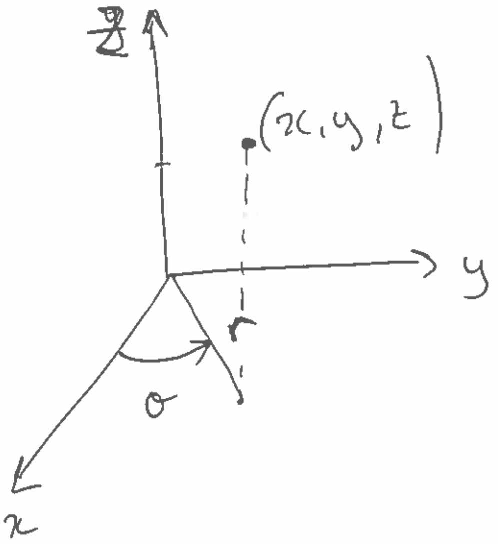


Triple integrals in cylindrical coordinates:

Many shapes that we want to integrate over have some sort of rotational symmetry (about an axis, or about the origin). We have 2 special coordinate systems designed to make these kind of integrals easier: cylindrical coords and spherical coords.

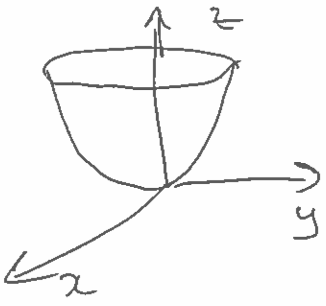
=> cylindrical coords: basically polar coords in xy with z: so (x,y,z) with (x,y) in polar coords (r,θ)

$$(x,y,z) \longleftrightarrow (r,\theta,z)$$

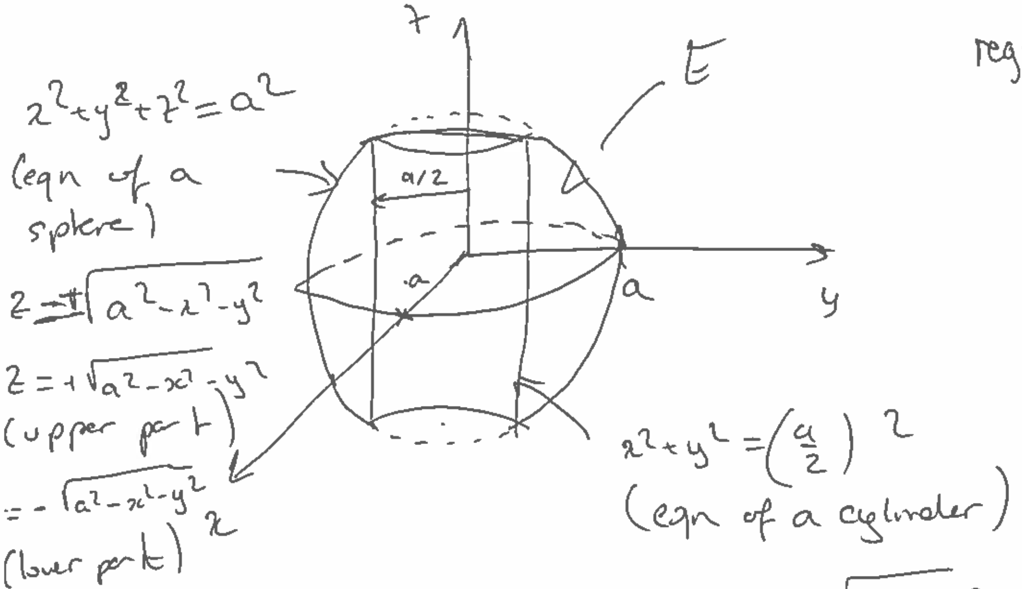


$$\Rightarrow \begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \\ z &= z \end{aligned}$$

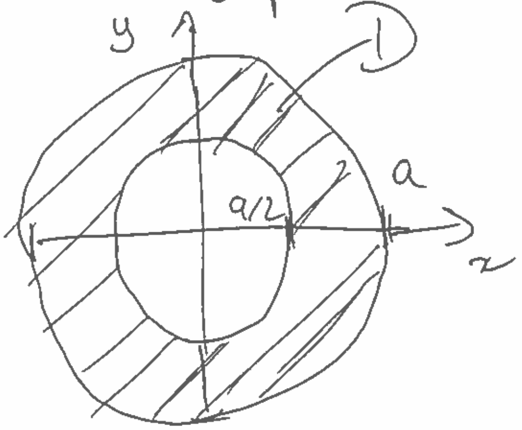
example: paraboloid $z = x^2+y^2$ in cylindrical coords is just $z = r^2$ (absence of theta, rotational symmetry)



Example: a drill bit of diameter a is used to drill a hole through a sphere of radius a. Find the volume of what remains.



region in xy plane



annulus $a/2 \le r \le a$

$0 \le \theta \le 2\pi$

So volume $V = \iiint_E dV = \iint \int_{-\sqrt{a^2-x^2-y^2}}^{+\sqrt{a^2-x^2-y^2}} dz\,dA$

$$= V = \iint_D 2\sqrt{a^2-x^2-y^2}\,dA = \int_0^{2\pi}\int_{a/2}^{a} 2\sqrt{a^2-r^2}\,r\,dr\,d\theta$$

$u = a^2 - r^2 \Rightarrow du = -2r\,du$

$r = a/2 \Rightarrow u = a^2 - \frac{a^2}{4} = \frac{3}{4}a^2$

$r = a \Rightarrow u = 0$

$$\int_{a/2}^{a} 2\sqrt{a^2-r^2}\, r\,dr = -\int_{3/4a^2}^{0} \sqrt{u}\,du = \int_{0}^{3/4a^2} u^{1/2}\,du$$

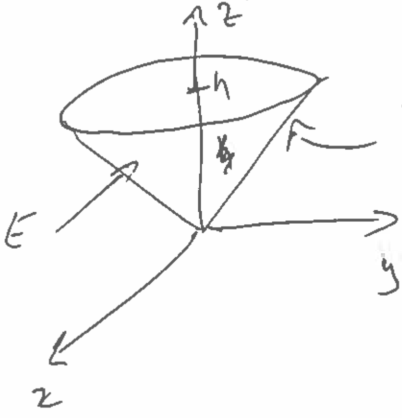
$$= \left[\frac{2}{3}u^{3/2}\right]_0^{3/4a^2} = \frac{2}{3}\,\frac{3^{3/2}}{4^{3/2}}\,a^3 = \frac{2\,\cancel{3}\sqrt{3}\,a^3}{\cancel{3}\,4\sqrt{4}} = \frac{\sqrt{3}\,a^3}{2\sqrt{4}}$$

$$= \frac{\sqrt{3}\,a^3}{4}$$

$$\text{so } V = \frac{\sqrt{3}a^3}{4}\int_0^{2\pi} d\theta = \boxed{\frac{\sqrt{3}}{2}\pi a^3}$$

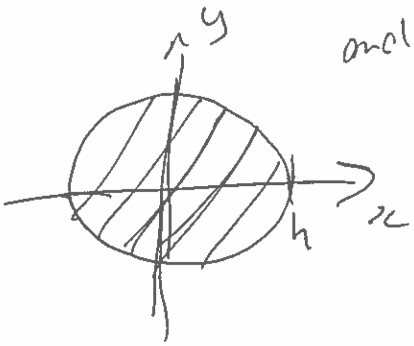
Example: Find the centre of mass $(\bar{x}, \bar{y}, \bar{z})$ of a right circular cone of height h:

=> easier to turn cone upside down (because it has a nice description in ~~polar~~ cylindrical coords)



$z = \sqrt{x^2+y^2} = r$

by symmetry $\bar{x} = \bar{y} = 0$



and $$\bar{z} = \frac{\iiint z\,dV}{\iiint_E dV}$$

Volume $V = \iiint_E dV = \iint_D \int_r^h dz\, dA = \int_0^{2\pi}\int_0^h (h-r) r\, dr\, d\theta$

$$= \int_0^{2\pi}\int_0^h (hr - r^2)\, dr\, d\theta = \int_0^{2\pi} \left[\frac{hr^2}{2} - \frac{r^3}{3}\right]_0^h d\theta = \int_0^{2\pi} \left(\frac{h^3}{2} - \frac{h^3}{3}\right) d\theta$$

$$V = 2\pi \frac{h^3}{6} = \boxed{\frac{\pi h^3}{3} = V}$$

$$\Rightarrow \int_0^{2\pi}\int_0^h\int_r^h z\, dz\, r\, dr\, d\theta = \int_0^{2\pi}\int_0^h \left[\frac{z^2}{2}\right]_r^h r\, dr\, d\theta$$

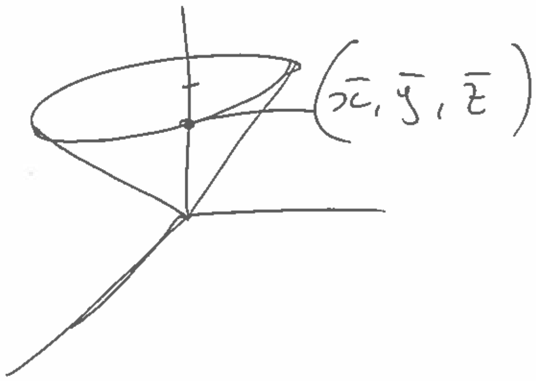
$$= \int_0^{2\pi}\int_0^h \left(\frac{h^2}{2} - \frac{r^2}{2}\right) r\, dr\, d\theta = \int_0^{2\pi}\int_0^h \left(\frac{h^2 r}{2} - \frac{r^3}{2}\right) dr\, d\theta$$

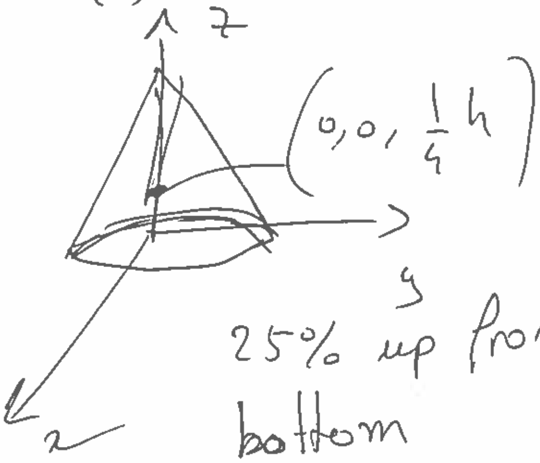
$$= \int_0^{2\pi} \left[\frac{h^2 r^2}{4} - \frac{r^4}{8}\right]_0^h d\theta = \int_0^{2\pi} \left(\frac{h^4}{4} - \frac{h^4}{8}\right) d\theta = 2\pi \frac{h^4}{8}$$

$$= \frac{\pi h^4}{4}$$

$$\text{So } \bar{z} = \frac{\pi h^4}{4} \Big/ \frac{\pi h^3}{3} = \frac{\cancel{\pi} h^4 \, 3}{4 \cancel{\pi} h^3} = \frac{3h}{4} = \bar{z}$$

$\Rightarrow$ so centre of gravity is $\left(0, 0, \frac{3h}{4}\right)$





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