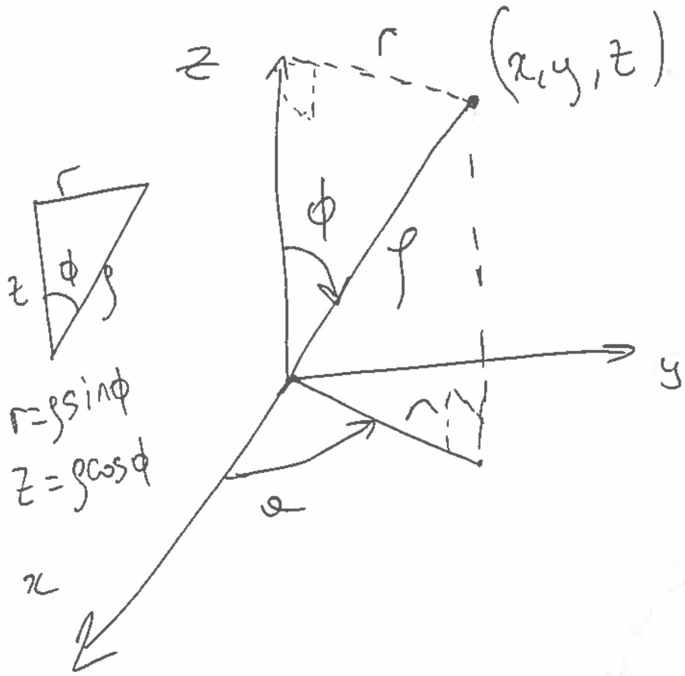
Spherical coordinates

Lecture 34

①



$x = \rho \sin\phi \cos\theta$

$y = \rho \sin\phi \sin\theta$

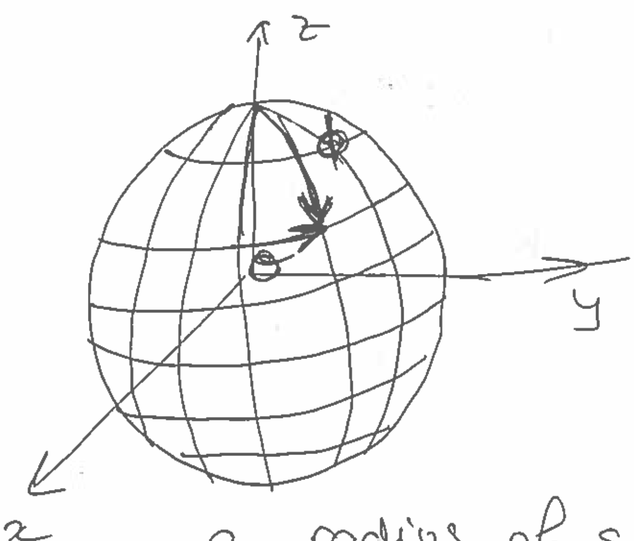
$z = \rho \cos\phi$

⚠ Some books/people interchange θ and ϕ

Interpretation as latitude and longitude.

North pole $\Rightarrow \phi = 0$

South pole $\Rightarrow \phi = \pi$



$0 \leq \rho \leq \infty$

$0 \leq \theta \leq 2\pi$

$0 \leq \phi \leq \pi$

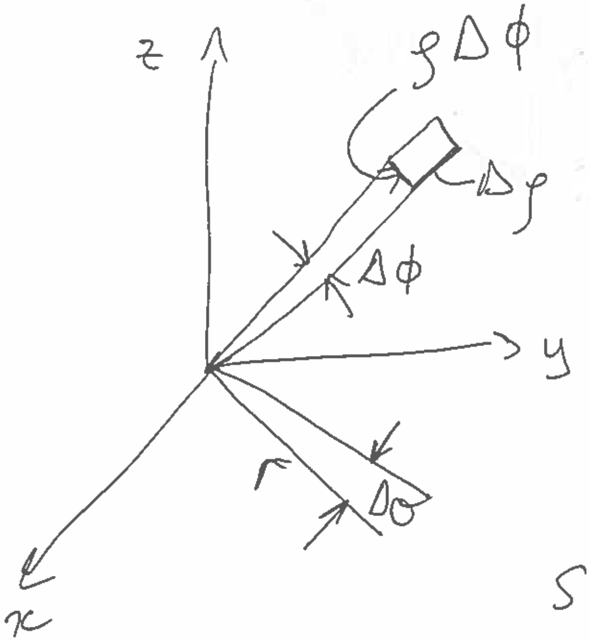
ρ = radius of sphere

Goal: write $\iiint_E f\, dV$ in spherical (ρ, θ, ϕ) rather than rectangular coordinates (x, y, z)

⇒ useful when E, f have a nice description

in spherical coords, e.g E is a sphere (or part of) $f(x,y,z) = \sqrt{x^2+y^2+z^2} = \rho$

Volume element: subdivide ρ, θ, ϕ into small intervals $\Delta\rho, \Delta\theta, \Delta\phi$ ⇒ ΔV ?



$$\Delta V = (\rho\,\Delta\phi)(\Delta\rho)(\overset{\rho\sin\phi}{r}\,\Delta\theta)$$

$$\Delta V = \rho^2 \sin\phi\,\Delta\phi\,\Delta\rho\,\Delta\theta$$

$$\text{so } \iiint_E f\,dV = \iiint_E f\,\rho^2 \sin\phi\,d\phi\,d\rho\,d\theta$$

works especially well when E is a ball (or part of a ball)

e.g $B = \{(x,y,z) : x^2+y^2+z^2 \leq 1\} = \{(\rho,\phi,\theta),\ 0 \leq \rho \leq 1,\ 0 \leq \phi \leq \pi,\ 0 \leq \theta \leq 2\pi\}$

Example: Evaluate $I = \iiint_B e^{(x^2+y^2+z^2)^{3/2}} \, dV$ 3

$\Rightarrow x^2+y^2+z^2 = \rho^2$

$$I = \int_{\theta=0}^{\theta=2\pi} \int_{\phi=0}^{\phi=\pi} \int_{\rho=0}^{\rho=1} e^{\rho^3} \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

$$I = \int_{\theta=0}^{\theta=2\pi} d\theta \int_{\phi=0}^{\phi=\pi} \sin\phi \, d\phi \int_{\rho=0}^{\rho=1} e^{\rho^3} \rho^2 \, d\rho$$

$\Rightarrow \int_{\rho=0}^{\rho=1} e^{\rho^3} \rho^2 d\rho \Rightarrow u = \rho^3 \Rightarrow du = 3\rho^2 d\rho$

$\Rightarrow d\rho = \frac{du}{3\rho^2}$

$\rho = 0 \Rightarrow u = 0$

$\rho = 1 \Rightarrow u = 1$

$$\int_{\rho=0}^{\rho=1} e^{\rho^3} \rho^2 d\rho = \int_{u=0}^{u=1} \frac{1}{3} e^u \, du = \frac{1}{3}\left(e - 1\right)$$

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$$\int_{\phi=0}^{\phi=\pi} \sin\phi \, d\phi = \Big[-\cos\phi\Big]_0^{\pi} = -(-1-1) = 2$$

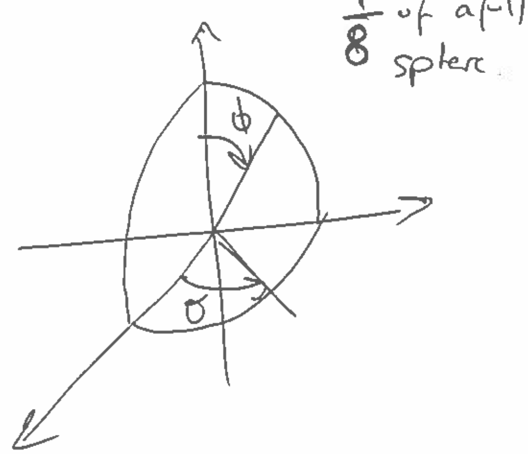
$$\int_{\theta=0}^{\theta=2\pi} d\theta = 2\pi$$

$$\Rightarrow \boxed{I = \frac{4\pi}{3}(e-1)}$$

Compare (side exercice) $I = \displaystyle\int_{x=-1}^{x=1}\int_{y=-\sqrt{1-x^2}}^{y=+\sqrt{1-x^2}}\int_{-\sqrt{1-x^2-y^2}}^{\sqrt{1-x^2-y^2}} e^{(x^2+y^2+z^2)^{3/2}} \, dz\,dy\,dx$

<u>example</u>: Let $E: \{(x,y,z) : x,y,z \geq 0 \,,\ x^2+y^2+z^2 \leq 9\}$

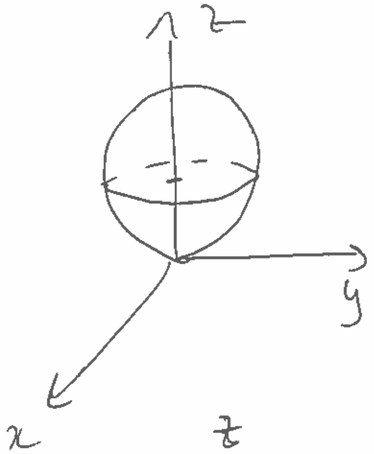
Write $I = \iiint_E f\,dV$ in spherical coords.



$E = \{(\rho,\phi,\theta) : 0 \leq \rho \leq 3 \,;\ 0 \leq \theta \leq \pi/2$

$0 \leq \phi \leq \pi/2\}$

Example Same as above for $E: \{(x,y,z) : x^2+y^2+(z-1)^2 \leq 1\}$

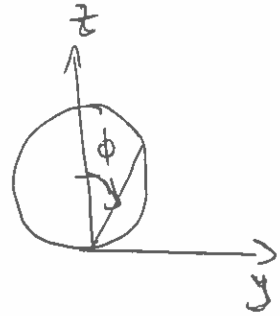


$$x^2+y^2+(z-1)^2 = x^2+y^2+z^2-2z+1 \leq 1$$

$$\Rightarrow x^2+y^2+z^2 \leq 2z$$

$$\rho^2 \leq 2\rho\cos\phi$$

$$\rho \leq 2\rho\cos\phi$$ not π ⚠



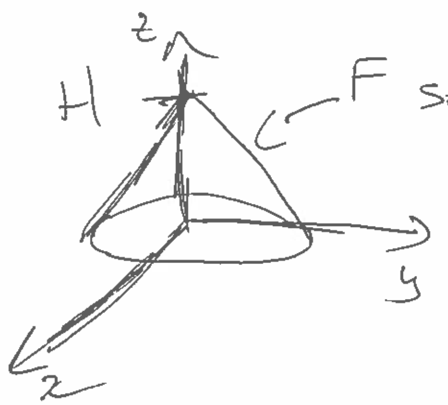
$$I = \int_0^{2\pi}\int_0^{\pi/2}\int_0^{2\rho\cos\phi} f \,\rho^2 \sin\phi \,d\rho\, d\phi\, d\theta$$

Example: How much work was done to raise Mt Fuji from sea-level?

Assume: solid circular cone with radius $R = 19{,}000\,m$
height $H = 4{,}000\,m$
density $\rho = 3000\,kg/m^3$

$\Rightarrow$ work to raise volume ΔV to height z is $Fd = \rho \Delta V g z$ [J]

$g = 9.8\,m/s^2$



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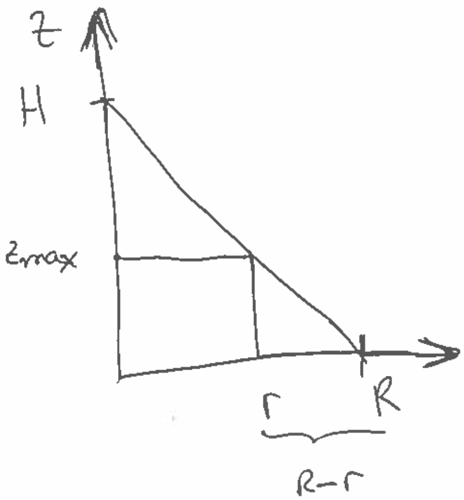
$$W = \text{Total work} = \iiint_F \rho g z \, dV = \rho g \iiint_F z \, dV$$



$$W = \rho g \int_0^{2\pi}\int_0^{R}\int_0^{\frac{H}{R}(R-r)} z \, dz \, r \, dr \, d\theta$$

similar triangles $\frac{R}{H} = \frac{R-r}{z_{max}}$

$$z_{max} = \frac{H}{R}(R-r)$$

$$\Rightarrow W = \rho g \int_0^{2\pi}\int_0^{R} \left[\frac{z^2}{2}\right]_0^{\frac{H}{R}(R-r)} r \, dr \, d\theta = \rho g \int_0^{2\pi}\int_0^{R} \frac{(HR-Hr)^2}{2R^2} r \, dr \, d\theta$$

$$W = \rho g \iint \frac{1}{2R^2}\left(H^2R^2 - 2H^2Rr + H^2r^2\right) r \, dr \, d\theta = \frac{\rho g}{2}\frac{H^2}{R^2} \underbrace{\int_0^{2\pi} d\theta}_{2\pi} \int_0^{R} R^2 r - 2Rr^2 + r^3 \, dr$$

$$= \frac{\rho g H^2}{\cancel{2} R^2} \cancel{2}\pi \left[\frac{R^2 r^2}{2} - \frac{2Rr^3}{3} + \frac{r^4}{4}\right]_0^R$$

$$= \frac{\rho g H^2 \pi}{R^2} \left[\frac{R^4}{2} - \frac{2R^4}{3} + \frac{R^4}{4}\right] = \frac{\rho g H^2 \pi}{R^2 12}\left(6R^4 - 8R^4 + 3R^4\right)$$

$$W = \frac{\rho g H^2 \pi R^4}{R^2 12} = \frac{\rho g H^2 \pi R^2}{12}$$

$$= \frac{\pi}{12} \times 3\times10^3 \times 16\times10^6 \times 19^2 \times 10^6 \times 9{,}8 = 4{,}45 \times 10^{19} [J]$$

for comparison: – Tsar Bomba : 2×10^{17} J
(wikipedia)

– All nukes $2{,}5\ 10^{19}$ J
(one half of Mt Fuji). see nuclear-darkness.org.

– Yerly ~~usa~~ energy usage in the USA
$9.4\ 10^{19}$ J (2 Mt Fujis).

– 2004 Tsunami earthquake (total)
10^{22} J (200 fujis).

– Energy in 100 trilions (10^{18}) Samsung Note 7's
$\Rightarrow$ One Fuji

– Energy to lift Trump's puny little wall
(3200km × 15 × 0,25m): less than one
millionth of a Fuji. Weak! and Sad.