Name (print):

Student number:

Section *(Please circle one)*: 001 002 003 004

University of British Columbia

MATH 110: MIDTERM TEST 1

Date: *October 16, 2013*
Time: *6:00 p.m. to 7:30 p.m.*
Number of pages: *9 (including cover page)*
Exam type: *Closed book*
Aids: *No calculators or other electronic aids*

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Question	Mark	Possible marks
1		7
2		4
3		5
4		6
5		4
6		8
7		6
Total		40

1. Let P be the point $(1, 2)$ and C be the circle described by the equation

$$(x-5)^2 + (y-5)^2 = 25.$$

All three parts of this question refer to the point P and the circle C.

2 marks (a) Does the point P lie on the circle C? Justify your answer.

Solution: To see if the point is on the curve, we substitute point P into the equation to see if it is true:

$$\begin{aligned}\text{LHS} &= (x-5)^2 + (y-5)^2 \\ &= ((1)-5)^2 + ((2)-5)^2 \\ &= (-4)^2 + (-3)^2 \\ &= 16 + 9 \\ &= 25 = \text{RHS}\end{aligned}$$

So that means the point satifies the equation and hence is on the curve.

2 marks (b) Find the equation of the line that passes through the point P and the centre of the circle C.

Solution: We know that $P = (1, 2)$. To get the equation of the straight line, we need the coordinates of the center of the circle. From the equation given, we can just read off that the center is at $(5, 5)$. So that we can get the slope:

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{5-2}{5-1} = \frac{3}{4}$$

We can then use the point slope formula:

$$y - y_1 = m(x - x_1)$$

along with the slope and the point $(5, 5)$

$$y - 5 = \frac{3}{4}(x-5)$$

This form will be used later on. But for completeness, we get that:

$$y = \frac{3}{4}x + \frac{5}{4}$$

3 marks (c) The line whose equation you found in part (b) intersects the circle C at exactly two points. Find the coordinates of those points. *Hint: Draw a picture.*

Solution: If we draw a picture, we see that the line we've found in he previous part is really a diameter. Making use of the rotational symmetry of the circle, we can find the other intersection point. The point P is 4 units left of the center and 3 units below. So the diameteric point will be 4 units right of the center and 3 units above. Since the center is at $(5, 5)$, we have that the other intersection point happens at $(5 + 4, 5 + 3) = (9, 8)$.

Solution: I suspect that most people will proceed to crunch through the algebra. To do that, we will make use of the two equations:

$$\begin{aligned} y - 5 &= \frac{3}{4}(x - 5) && \text{prev. part} \\ (x-5)^2 + (y-5)^2 &= 25 && \text{Given} \end{aligned}$$

if we substitute the first into the second

$$\begin{aligned} (x-5)^2 + \left(\frac{3}{4}(x-5)\right)^2 &= 25 \\ (x-5)^2 + \left(\frac{3}{4}\right)^2 (x-5)^2 &= 25 \\ \left(\left(\frac{3}{4}\right)^2 + 1\right)(x-5)^2 &= 25 \\ \left(\frac{9}{16} + 1\right)(x-5)^2 &= 25 \\ \left(\frac{25}{16}\right)(x-5)^2 &= 25 \\ (x-5)^2 &= 25\left(\frac{16}{25}\right) \\ (x-5)^2 &= 16 \\ x - 5 &= \pm 4 \end{aligned}$$

This means $x = 1, 9$. When $x = 1$, we can substitute it into either equation to get $y = 2$. This is our original point. The alternative solution is when $x = 9$ and we get $y = 8$ to give $(9, 8)$ as the other intersection point.

2. Both parts of this question refer to the functions

$$f(x) = \sqrt{2x+1} \text{ and } g(x) = \begin{cases} -1 & \text{if } x < 0 \\ 0 & \text{if } x = 0 \\ 1 & \text{if } x > 0 \end{cases}.$$

2 marks (a) On its entire domain, the function $(f \circ g)(x)$ has a *range*, or output, of exactly two numbers. Find those numbers, and explain your answers.

Solution: Let's consider the output of $g(x)$, we see that for any input x, we have the output to be $-1, 0, 1$. That means those 3 values are the only possible inputs going into $f(x)$. We see that $f(-1)$ is not defined. So we are left with $f(0) = \sqrt{2(0)+1} = \sqrt{1}$ and $f(1) = \sqrt{2(1)+1} = \sqrt{3}$. So the only two outputs we have for $(f \circ g)(x) = 1, \sqrt{3}$.

2 marks (b) On its entire domain, the function $(g \circ f)(x)$ also has a range of exactly two numbers. Find those numbers, and explain your answers.

Solution: We play the same game here. For any x in the domain of f, we have that $f(x) \geq 0$ due to the square root. So that means the only inputs for g that we have to consider are those that are 0 or positive. We know $g(0) = 0$ and for any value $x > 0$, we have $g(x) = 1$. So that only outputs of $(g \circ f)(x) = 0, 1$.

3. Suppose the position of a particle with respect to time is described by the function

$$p(t) = 2t^2 - 3t + 4,$$

where position is measured in metres and time is measured in seconds. All three parts of this question refer to this particle.

2 marks (a) Calculate the average velocity of the particle between $t = 2$ seconds and $t = 3$ seconds.

Solution: We recognise that average velocity is really just the slope of the function between the two points. So we use:

$$\begin{aligned} v_{\text{ave}} &= \frac{p(3) - p(2)}{3 - 2} \\ &= \frac{(2(3)^2 - 3(3) + 4) - (2(2)^2 - 3(2) + 4)}{3 - 2} \\ &= \frac{(18 - 9 + 4) - (8 - 6 + 4)}{1} \\ &= (13) - (6) \\ &= 7 \end{aligned}$$

The average velocity between times $t = 2$ and $t = 3$ is 7 (metres per seconds)

2 marks (b) Calculate the average velocity of the particle between $t = 2$ seconds and $t = 2 + h$ seconds, where h is a small positive number.

Solution: We repeat the same calculation as before, except with $t = 2 + h$ in place of $t = 3$

$$\begin{aligned} v_{\text{ave}} &= \frac{p(2+h) - p(2)}{2 + h - 2} \\ &= \frac{(2(2+h)^2 - 3(2+h) + 4) - (2(2)^2 - 3(2) + 4)}{(2 + h - 2} \\ &= \frac{(2(4 + 4h + h^2) - 6 - 3h + 4) - (6)}{h} \\ &= \frac{(8 + 8h + 2h^2 - 2 - 3h) - (6)}{h} \\ &= \frac{(5h + 2h^2 + 6) - (6)}{h} \\ &= \frac{5h + 2h^2}{h} \\ &= \frac{h(5 + 2h)}{h} \\ &= 5 + 2h. \end{aligned}$$

So the average velocity between time $t = 2$ and time $t = 2 + h$ is $5 + 2h$ (metres per second).

1 mark (c) Estimate the instantaneous velocity of the particle at $t = 2$ seconds, and explain your answer.

Solution: This question is really asking us to take a limit as the distance between 2 and $2 + h$ goes to 0. This means we are taking a limit as $h \to 0$.

$$\begin{aligned} v(2) &= \lim_{h \to 0} \frac{p(2+h) - p(2)}{2 + h - 2} \\ &= \lim_{h \to 0} 5 + 2h && \text{Prev. part.} \\ &= 5 && \text{Direct substitution.} \end{aligned}$$

So the velocity at $t = 2$ is approximately 5 (metres per second).

4. Evaluate each of the following limits or show that the limit does not exist.

2 marks (a) $\lim_{x\to 4} 2x^2 - 5x - 12$

Solution: We recognise that the expression we are trying to take a limit of is a polynomial. It is continuous, therefore we can directly substitute.

$$\lim_{x\to 4} 2x^2 - 5x - 12 = 2(4)^2 - 5(4) - 12 = 32 - 20 - 12 = 0$$

Therefore, the limit we get is 0.

2 marks (b) $\lim_{x\to 4} \frac{2x^2 - 5x - 12}{x-4}$

Solution: From the previous part, we know that the numerator will result in 0 if we directly substitute and the denominator will as well. So that means we'll have to do more work. We try factoring the top (either through observation, quadratic formula, or long division) to get that:

$$\begin{aligned}
\lim_{x\to 4} \frac{2x^2 - 5x - 12}{x-4} &= \lim_{x\to 4} \frac{(2x+3)(x-4)}{x-4} \\
&= \lim_{x\to 4}(2x+3) && \text{Simplify} \\
&= (2(4)+3) && \text{Direct Substutition.} \\
&= 11.
\end{aligned}$$

So we have the limit of this expression will be 11.

2 marks (c) $\lim_{x\to 4} \frac{2x^2 - 5x - 12}{|x-4|}$

Solution: We play the same games as per the previous part. We see that direct substitution will result in "$\frac{0}{0}$". So we try factoring to get:

$$\lim_{x\to 4} \frac{2x^2 - 5x - 12}{|x-4|} = \lim_{x\to 4}(2x+3) \cdot \frac{(x-4)}{|x-4|}$$

It looks like we cannot simplify anymore. But luckily, it also looks like something we've seen in class previous. Let's consider the two one-sided limits to see what happens:

- For $\lim_{x\to 4^-}(2x+3) \cdot \frac{(x-4)}{|x-4|}$, we consider numbers close to 4 but less than it. So that means $x - 4 < 0$ and $|x-4| > 0$. That means the fraction will be negative and the overall limit will yield -11.

- For $\lim_{x\to 4^+}(2x+3) \cdot \frac{(x-4)}{|x-4|}$, we consider numbers close to 4 but greater than it. So that means $x - 4 > 0$ and $|x - 4| > 0$. That means the fraction will be positive and the overall limit will yield 11.

Since the two one-sided limits are not equal, the overall limit cannot exist. That is:

$$\lim_{x \to 4} \frac{2x^2 - 5x - 12}{|x - 4|} \quad \text{DNE}$$

4 marks 5. Show that the function $f(x) = x^3 - 3x^2 + 1$ has at least three roots in the interval $[-1, 3]$.

Solution: Since $f(x)$ is a polynomial, it must be continuous on the entire real line. If we computed $f(-1) = -3$ and $f(3) = 1$, we can use the IVT to show that there is at least one root between $[-1, 3]$. However, we want to show the existence of at least 3. This suggests that we should refine our search. Computing a table of values, we get that:

x	-1	0	1	2	3
f(x)	-3	1	-1	-3	1

From this, we can see that we want to apply the IVT to three separate intervals: $[-1, 0]$, $[0, 1]$ and $[2, 3]$. These three intervals do not overlap at all, so the roots we get by applying the IVT to each of them will be distinct roots. (The only overlap is the point $x = 0$, but we can see that $f(0) \neq 0$ and hence is not the root we are looking for.)

Thus, we get a distinct root for each of the intervals $[-1, 0]$, $[0, 1]$ and $[2, 3]$. This gives us the three roots that we're looking for.

[**1 bonus mark**] Explain why $f(x)$ cannot have more than three roots in the interval $[-1, 3]$.

Solution: To say that a cubic has at most three roots would not be enough for this question because we have not shown why a degree n polynomial can have at most n roots. The simplest method is something called "proof by contradiction". The basis of this method is to assume that it has more than 3 roots and see what can go wrong. So let's suppose $f(x)$ has more than three roots. Let's label four of the roots A, B, C, D. Then we must have

$$f(x) = (x - A)(x - B)(x - C)(x - D)g(x)$$

for some other polynomial $g(x)$. We get those factors because they ensure that $f(x) = 0$ at the correct values. By expanding the right hand side, we get that

$$f(x) = (x^4 + \ldots)g(x).$$

In particular, $f(x)$ must have a term that is starts with x^4. This would contradict that $f(x)$ is a cubic because the highest power is no longer x^3. This means our initial assumption of having more than three roots must be incorrect. Hence $f(x)$ cannot have more than three roots.

6. Determine whether each of the following statements is true or false. If it is true, provide justification. If it is false, provide a counterexample.

2 marks (a) The graph of $f(x) = \frac{1}{2}\left((x+3)^2 - 2\right)$ crosses the x-axis.

Solution: **True:** There are multiple ways to see this.

- Solve the quadratic equation. The function expands to: $f(x) = \frac{1}{2}x^2 + 3x + \frac{7}{2}$. Which then gives:

$$\begin{aligned} x &= \frac{-3 \pm \sqrt{9 - 4\left(\frac{1}{2}\right)\left(\frac{7}{2}\right)}}{2\left(\frac{1}{2}\right)} \\ &= \frac{-3 \pm \sqrt{9-7}}{1} \\ x &= -3 \pm \sqrt{2} \end{aligned}$$

- We can also use the vertex formula by seeing that $f(x) = \frac{1}{2}(x+3)^2 - 1$, with vertex $(-3, -1)$ which is below the axis. Combining the the positive coefficient of $\frac{1}{2}$, meaning that the parabola is curving upwards, we get that it must cross the x axis. This also requires that a parabola be continuous (which it is cause its a polynomial).
- We can view this as a graph transformation of $y = x^2$ which touches the x-axis. Then we shift it 3 to the left and then 2 down, so it still touches the x-axis. This is then followed by a vertical compression of 2 (or expansion by 0.5). This does not change direction of curvature of the function and so it must still touch the x-axis.
- We can also use the IVT. The function is a quadratic and hence is continuous. We then look at $f(-3) = -1 < 0$ and then $f(0) = 3.5 > 0$. So there must be a point between -3 and 0 that crosses the x-axis.

2 marks (b) If $\lim_{x \to 1} f(x) = 4$, then $f(1) = 4$.

Solution: **False:** The easiest way to see this is to come up with a function where the limit exists but the function is not defined there. For example $f(x) = \frac{4(x-1)}{x-1}$. The limit exists and is equal to 4 but 1 is not in the domain, so $f(1)$ is not defined.

2 marks (c) If $f\left(\frac{1}{10}\right) = 2$, $f\left(\frac{1}{100}\right) = 2$, $f\left(\frac{1}{1000}\right) = 2$, and in fact $f\left(\frac{1}{10^n}\right) = 2$ for every integer n, then $f(0) = 2$.

Solution: **False:** We can use the same function as above to show that we can have every point leading to 0 to be defined at 2 but $f(0)$ to be undefined. For example: $f(x) = \frac{2x}{x}$. We have $f(x) = 2$ for all x except for 0 and $f(0)$ to be undefined.

2 marks (d) $f(x) = \begin{cases} \frac{3}{x+2} & \text{if } x < 1 \\ \sqrt{x} & \text{if } x \geq 1 \end{cases}$ is continuous at all real numbers.

Solution: False: To hunt for discontinuities, we have to check inside each piece and also at the boundary.

- For the boundary to be continuous, we must have:

$$\begin{aligned} \lim_{x\to1^-} f(x) &= \lim_{x\to1^+} f(x) \\ \lim_{x\to1^-} \frac{3}{x+2} &= \lim_{x\to1^+} \sqrt{x} \\ \frac{3}{1+2} &= \sqrt{1} && \text{Direct Substitution.} \\ 1 &= 1. \end{aligned}$$

  So the limits exists and is equal to 1 which happens to be $f(1) = 1$. So the function in continuous at the boundary.

- On the right of $x = 1$, the function is continuous since $\sqrt{x}$ is only undefined for $x < 0$ which is not covered by this case.
- On the left of $x = 1$, the function is discontinuous at $x = -2$ since the denominator is 0. This fall inside the region considered. So $f(x)$ is discontinuous at $x = -2$.

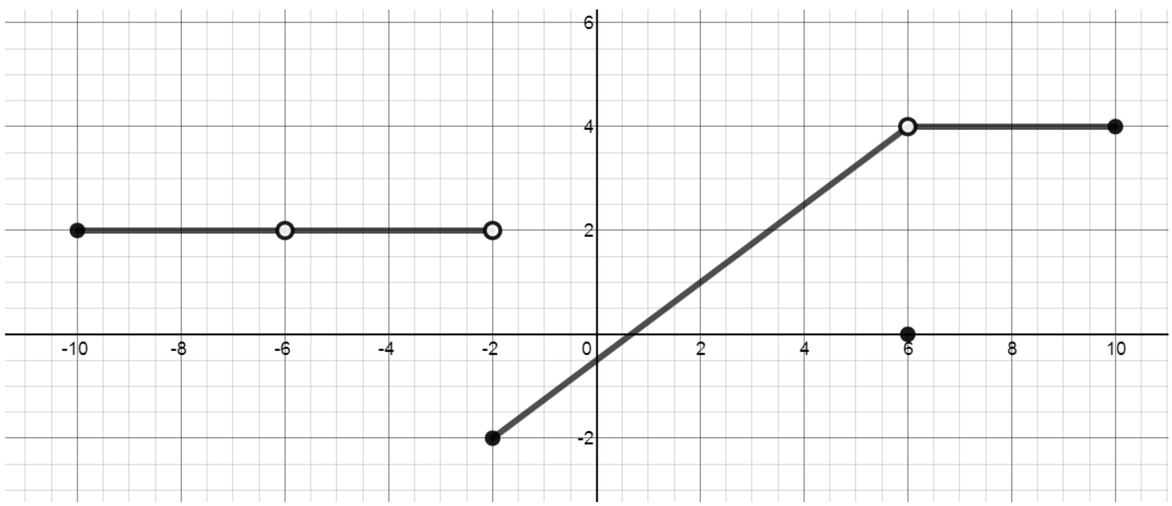
That means $f(x)$ is NOT continuous over all real numbers.

6 marks 7. On the grid below, sketch a function $f(x)$ satisfying all of the following 6 properties:

- The domain is $[-10,-6) \cup (-6,10]$.
- For every value x in the domain, $-4 \leq f(x) \leq 6$.
- $\lim_{x \to 6} f(x) = 4$
- $f(6) = 0$
- $\lim_{x \to -2^-} f(x) = 2$
- $\lim_{x \to -2^+} f(x) = -2$

You are not required to come up with an algebraic equation for the function.

Solution: One sample function is below. It is not the only solution. I just used straight lines to make things easier.



This page may be used for rough work. It will not be marked.