

MATH 110 Midterm 1, October 25th, 2016**Duration: 90 minutes***This test has 7 questions on 10 pages, for a total of 75 points.*

- Read all the questions carefully before starting to work.
- All questions require a full solution; you should give complete arguments and explanations for all your calculations; answers without justifications will not be marked.
- Continue on the back of the previous page if you run out of space.
- Attempt to answer all questions for partial credit.
- This is a closed-book examination. **None of the following are allowed:** documents, cheat sheets or electronic devices of any kind (including calculators, cell phones, etc.)

First Name: _____ Last Name: _____

Student-No: _____ Section: _____

Signature: _____

Question:	1	2	3	4	5	6	7	Total
Points:	15	10	13	8	14	7	8	75
Score:								

Student Conduct during Examinations

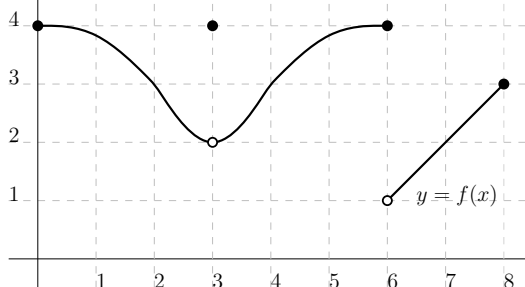
1. Each examination candidate must be prepared to produce, upon the request of the invigilator or examiner, his or her UBCcard for identification.
 - (ii) purposely exposing written papers to the view of other examination candidates or imaging devices;
2. Examination candidates are not permitted to ask questions of the examiners or invigilators, except in cases of supposed errors or ambiguities in examination questions, illegible or missing material, or the like.
 - (iii) purposely viewing the written papers of other examination candidates;
3. No examination candidate shall be permitted to enter the examination room after the expiration of one-half hour from the scheduled starting time, or to leave during the first half hour of the examination. Should the examination run forty-five (45) minutes or less, no examination candidate shall be permitted to enter the examination room once the examination has begun.
 - (iv) using or having visible at the place of writing any books, papers or other memory aid devices other than those authorized by the examiner(s); and,
 - (v) using or operating electronic devices including but not limited to telephones, calculators, computers, or similar devices other than those authorized by the examiner(s)(electronic devices other than those authorized by the examiner(s) must be completely powered down if present at the place of writing).
4. Examination candidates must conduct themselves honestly and in accordance with established rules for a given examination, which will be articulated by the examiner or invigilator prior to the examination commencing. Should dishonest behaviour be observed by the examiner(s) or invigilator(s), pleas of accident or forgetfulness shall not be received.
 6. Examination candidates must not destroy or damage any examination material, must hand in all examination papers, and must not take any examination material from the examination room without permission of the examiner or invigilator.
5. Examination candidates suspected of any of the following, or any other similar practices, may be immediately dismissed from the examination by the examiner/invigilator, and may be subject to disciplinary action:
 - (i) speaking or communicating with other examination candidates, unless otherwise authorized;
 7. Notwithstanding the above, for any mode of examination that does not fall into the traditional, paper-based method, examination candidates shall adhere to any special rules for conduct as established and articulated by the examiner.
 8. Examination candidates must follow any additional examination rules or directions communicated by the examiner(s) or invigilator(s).

Full-Solution Problems. In the following questions, justify your answers and **show all your work**. Unless otherwise indicated, **simplification of answers are required**.

1. This question has 4 different problems.

5 marks

(a) Consider the function $f(x)$ whose graph is shown below:



Find the following:

(i) domain of f : $[0, 8]$

(ii) range of f : $(1, 4]$

(iii) $\lim_{x \rightarrow 3} f(x) = 2$

slope of the line \leftarrow (iv) $f'(7) = \frac{\text{Rise}}{\text{Run}} = \frac{3-1}{8-6} = 1$

(v) $f(f(4)) = 4$

3 marks

(b) Find the domain of $f(x) = \frac{1}{\sqrt{x}-1}$.

Denominator $\neq 0 \rightarrow \sqrt{x} \neq 1 \Rightarrow x \neq 1$

Square Root: $x \geq 0$

\Rightarrow Domain: $x \geq 0$ and $x \neq 1$

or: $[0, 1) \cup (1, +\infty)$

3 marks

(c) Consider the functions $h(x) = \frac{x^2 + 1}{x}$ and

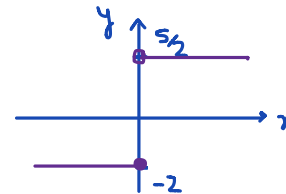
$$g(x) = \begin{cases} -1 & x \leq 0 \\ 2 & x > 0. \end{cases}$$

Find a formula for $h(g(x))$ and determine whether the composite function $h(g(x))$ is invertible. Justify your answer.

$$h(g(x)) = \begin{cases} \frac{(-1)^2 + 1}{-1} & x \leq 0 \\ \frac{2^2 + 1}{2} & x > 0 \end{cases} = \begin{cases} -2 & x \leq 0 \\ \frac{5}{2} & x > 0 \end{cases}$$

$h(g(x))$ NOT 1-1 \Rightarrow NOT invertible

you can also check the graph to
see that it fails the horizontal line test.



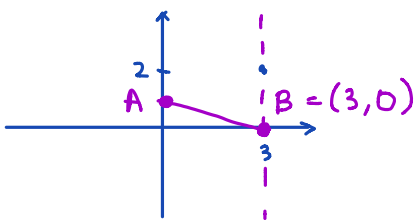
4 marks

(d) Let A be the y -intercept of the line that goes through the points $(-1, 2/3)$ and $(3, 2)$. Let B be the x -intercept of the line perpendicular to the x -axis that goes through the point $(3, 2)$. Find the distance between the points A and B .

$$m = \frac{2 - \frac{2}{3}}{3 - (-1)} = \frac{\frac{4}{3}}{4} = \frac{4}{12} = \frac{1}{3}$$

Choose one point: $y - 2 = \frac{1}{3}(x - 3) \Rightarrow y = \frac{1}{3}x + 1$

$$y\text{-int} : y = \frac{1}{3} \cdot 0 + 1 = 1 \Rightarrow A = (0, 1)$$



$$\Rightarrow B = (3, 0)$$

$$\begin{aligned} AB &= \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} \\ &= \sqrt{(3 - 0)^2 + (0 - 1)^2} \\ &= \sqrt{9 + 1} = \sqrt{10} \end{aligned}$$

10 marks

2. Evaluate the following limits, if they exist, or specify whether they are ∞ , $-\infty$, or do not exist. Show all your work.

$$(a) \lim_{x \rightarrow 5} \frac{x^2 - 1}{x - 1} = \frac{5^2 - 1}{5 - 1} = \frac{24}{4} = 6$$

$$(b) \lim_{t \rightarrow 0} \frac{t+1}{t^2} = \frac{0+1}{0^2} = \frac{1}{0^+} = +\infty$$

\downarrow always positive \downarrow always a small positive number

$$(c) \lim_{a \rightarrow 3^-} \frac{a^2 + 2a + 1}{a - 3} = \frac{3^2 + 2 \cdot 3 + 1}{3^- - 3} = \frac{16}{0^-} = -\infty$$

\downarrow a small negative number

$$(d) \lim_{x \rightarrow 2} \frac{x^2 - 6x + 8}{x^2 - 3x + 2} = \frac{2^2 - 6 \cdot 2 + 8}{2^2 - 3 \cdot 2 + 2} = \frac{0}{0} \rightarrow \text{Factorize}$$

$$= \lim_{x \rightarrow 2} \frac{(x-4)\cancel{(x-2)}}{\cancel{(x-2)}(x-1)} = \frac{2-4}{2-1} = -2$$

$$(e) \lim_{t \rightarrow -2} \frac{1+t^2}{2+t} = \frac{1+(-2)^2}{2+(-2)} = \frac{5}{0}$$

\downarrow positive or negative?

$$= \begin{cases} \lim_{t \rightarrow -2^-} \dots = \frac{5}{0^-} = -\infty \\ \lim_{t \rightarrow -2^+} \dots = \frac{5}{0^+} = +\infty \end{cases}$$

limit DNE

3. This question has 3 different problems.

4 marks

(a) Find y' if $y = 2x^5 + \sqrt[3]{x^2} - \frac{3}{\sqrt{x^3}} = 2x^5 + x^{\frac{2}{3}} - 3x^{-\frac{3}{2}}$

Power Rule: $y' = 10x^4 + \frac{2}{3}x^{\frac{2}{3}-1} - 3(-\frac{3}{2})x^{-\frac{3}{2}-1}$

$$= 10x^4 + \frac{2}{3}x^{-\frac{1}{3}} + \frac{9}{2}x^{-\frac{5}{2}}$$

Simplification is optional.

$$= 10x^4 + \frac{2}{3\sqrt[3]{x}} + \frac{9}{2x^2\sqrt{x}}$$

5 marks

(b) Find the equation of a line tangent to the graph of $y = x^3 - 3x$ at the point of x -coordinate $x = 2$.

point $x=2 \Rightarrow y = 2^3 - 3 \cdot 2 = 2 \Rightarrow (2, 2)$

slope $m = f'(2) = y'(2)$

$$y'(x) = 3x^2 - 3 \Rightarrow y'(2) = 3(2)^2 - 3 = 9$$

$$y - y_1 = m(x - x_1) \Rightarrow y - 2 = 9(x - 2)$$

$$\Rightarrow y = 9x - 16$$

4 marks

(c) Find a value for the constant a such that the graph of $y = ax^3$ is tangent to the line $y = 3x + 1$ at some point.

The tangency point is NOT given, but we know it's on f so we have the coordinates for that: $P = (x, ax^3)$

tangent line $y = \underset{m_{\text{tan}}}{3}x + 1$, but $m_{\text{tan}} = f'(x)$

$$3 = 3ax^2 \Rightarrow ax^2 = 1 \Rightarrow a = \frac{1}{x^2}$$

tangency point is both on line and on the graph \rightarrow it must satisfy both equations.

$$y = 3x + 1$$

$$y = ax^3 \rightarrow 3x + 1 = ax^3 = \frac{ax^2}{1} \cdot x$$

$$\Rightarrow 3x + 1 = x \Rightarrow 2x = -1 \Rightarrow x = -\frac{1}{2}$$

$$\Rightarrow a = \frac{1}{x^2} = \frac{1}{(-\frac{1}{2})^2} = \frac{1}{\frac{1}{4}} = 4$$

8 marks 4. Consider the function

$$f(x) = \begin{cases} x^2 - x - 6 & x < 0 \\ -6 & x > 0. \end{cases}$$

Answer the questions below. No marks will be given to answers without justification.

(a) Graph $f(x)$. Make sure you mark the intercepts of the function with the coordinate axes, if they exist, by indicating their value on the axes.

$$y = x^2 - x - 6$$

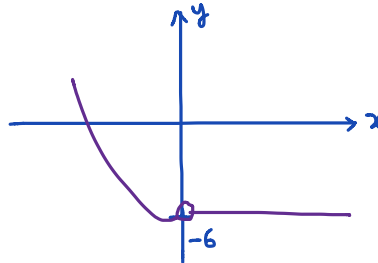
$$y\text{-int} : (0, -6)$$

$$x\text{-int} : x^2 - x - 6 = 0$$

$$(x-3)(x+2) = 0$$

$$x = 3, x = -2$$

$$\text{vertex} : x = \frac{-b}{2a} = \frac{1}{2},$$



(b) Is there a number a for which the limit $\lim_{x \rightarrow a} f(x)$ does not exist? Justify your answer.

Check left & right limits : (The only point to check is 0).

$$\lim_{x \rightarrow 0^-} f(x) = \lim_{x \rightarrow 0^-} x^2 - x - 6 = -6$$

$$\lim_{x \rightarrow 0^+} f(x) = -6$$

Equal \Rightarrow Limit exists everywhere \Rightarrow NO such "a" exists.
For limit we do NOT care about $f(0)$

(c) Is there a number a such that $f(x)$ is not continuous at $x = a$? Justify your answer.

$$\text{Check } \lim_{x \rightarrow 0^-} f(x) = \lim_{x \rightarrow 0^+} f(x) = f(0)$$

however, $f(0)$ is NOT defined in the equation, (Hole in the graph) \Rightarrow NOT cont's at $x=0$.

(d) Is there a number b such that the following piecewise function

$$g(x) = \begin{cases} f(x) & x < 0 \\ b & x = 0 \\ -6 + x & x > 0 \end{cases}$$

From above $f(x) = x^2 - x - 6$

is continuous everywhere? Justify your answer.

To have continuity everywhere

$$\lim_{x \rightarrow 0^-} g(x) = \lim_{x \rightarrow 0^+} g(x) = g(0)$$

$$\lim_{x \rightarrow 0^+} -6 + x = -6$$

$$\lim_{x \rightarrow 0^-} x^2 - x - 6 = -6 \Rightarrow g(0) = b = -6$$

14 marks

5. An outdoor hot tub holds 4000L of water. If a small valve at the bottom of the tub is opened, then the volume of water in the tub is modelled by the function

$$V(t) = 4000(1 - t)^2,$$

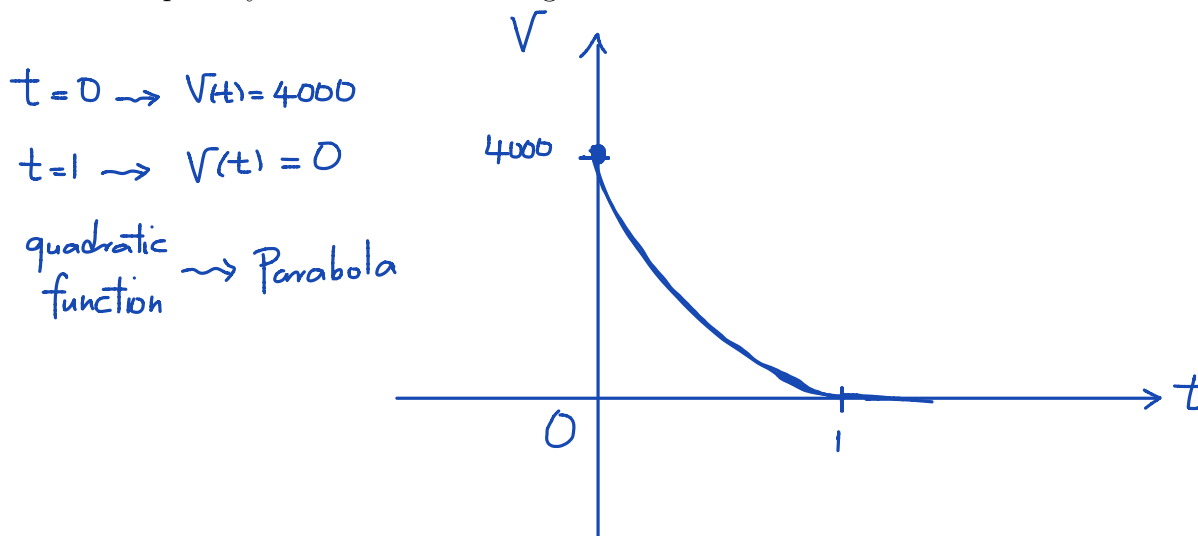
where V is the volume of water in the hot tub, in *litres*, and t is the time, in *hour*, since the valve is open.

- (a) How long does it take for the water to completely drain?

$$\underline{\underline{V(t) = 0}}$$

$$\begin{aligned} \Rightarrow 4000(1-t)^2 = 0 &\Rightarrow (1-t)^2 = 0 \\ &\Rightarrow t = 1 \text{ hour} \end{aligned}$$

- (b) Sketch the graph of the volume function. (Note that by part (a) there would be no water in the tub after some point in time.) Make sure you indicate the value of any intercept on your sketch. Your diagram does not need to be in scale.



- (c) Find the average rate of change in the volume of the water during the first half hour. Include units in your answer.

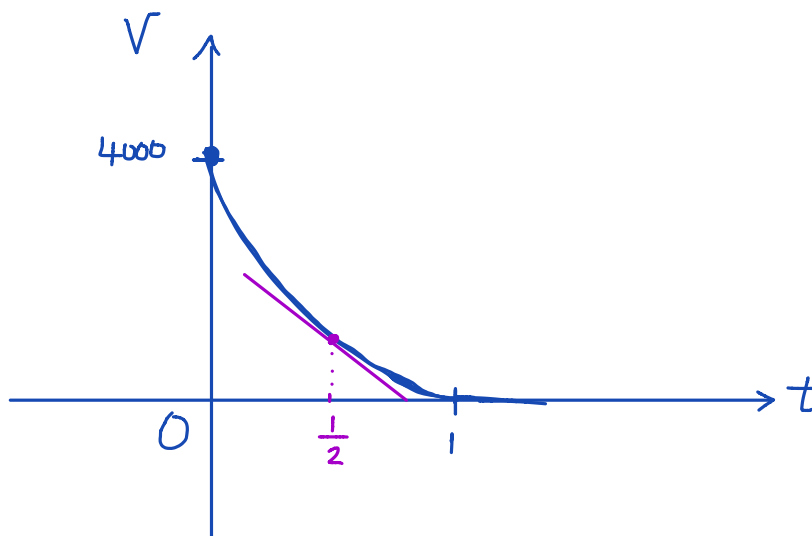
$$\underline{\underline{[0, \frac{1}{2}]}}$$

$$\begin{aligned} R_{av} &= \frac{V(\frac{1}{2}) - V(0)}{\frac{1}{2} - 0} \\ &= \frac{4000(1 - \frac{1}{2})^2 - 4000(1)^2}{\frac{1}{2}} \\ &= \frac{4000(\frac{1}{4} - 1)}{\frac{1}{2}} = 4000 \frac{-\frac{3}{4}}{\frac{1}{2}} = 4000 \cdot -\frac{3}{2} = -6000 \frac{\text{lit}}{\text{hour}} \end{aligned}$$

- (d) Using the definition of instantaneous rate of change as a limit, compute the instantaneous rate of change of V at $t = 1/2$ hour.

$$\begin{aligned}
 & \xrightarrow{\hspace{10em}} V'\left(\frac{1}{2}\right) \\
 V'\left(\frac{1}{2}\right) &= \lim_{h \rightarrow 0} \frac{V\left(\frac{1}{2}+h\right) - V\left(\frac{1}{2}\right)}{h} = \lim_{h \rightarrow 0} \frac{4000\left(1 - \left(\frac{1}{2}+h\right)\right)^2 - 4000\left(1 - \frac{1}{2}\right)^2}{h} \\
 &= \lim_{h \rightarrow 0} \frac{4000\left(\frac{1}{2} - h\right)^2 - 1000}{h} \\
 &= \lim_{h \rightarrow 0} \frac{4000\left(\frac{1}{4} - h + h^2\right) - 1000}{h} \\
 &= \lim_{h \rightarrow 0} \frac{1000 - 4000h + 4000h^2 - 1000}{h} \\
 &= \lim_{h \rightarrow 0} \frac{h(-4000 + 4000h)}{h} \\
 &= \underline{\underline{-4000}} \frac{\text{lit}}{\text{hour}}
 \end{aligned}$$

- (e) Determine the slope of the tangent line to the graph of $V(t)$ at $t = 1/2$ hour. Then sketch the tangent line on your graph created in part (b) (or reproduce the graph below).

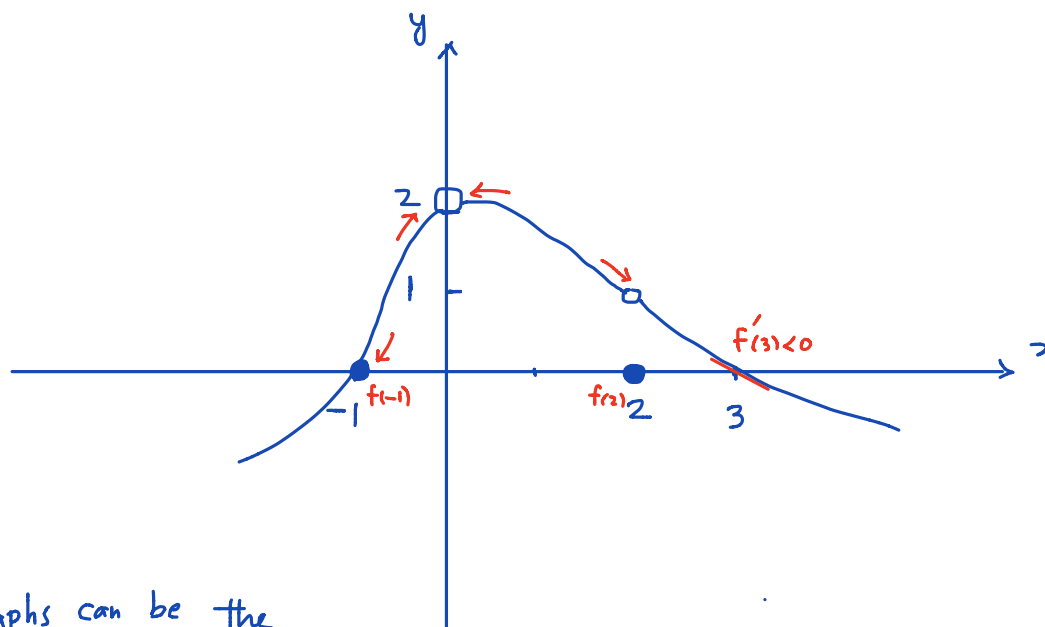


7 marks

6. Sketch the graph of a function f satisfying all of the following properties:

1. The domain of the function is all real numbers except 0,
2. $\lim_{x \rightarrow -1^+} f(x) = 0$,
3. $\lim_{x \rightarrow 0} f(x) = 2$,
4. $\lim_{x \rightarrow 2^-} f(x) = 1$,
5. $f(-1) = f(2)$,
6. $f'(3) < 0$.

Make sure you clearly identify the points on the graph that are related to the above conditions by indicating the value of their coordinates on the axes. Your graph does not need to be in scale. *Note:* You are NOT required to provide a formula for $f(x)$.



Different graphs can be the answer to this question.

8 marks

7. (a) State the Intermediate Value Theorem for a function f defined on an interval $[a, b]$.
Make sure you clearly state the assumptions and the conclusion of the theorem.

If f is continuous on $[a, b]$ and $f(a) \neq f(b)$, and if N is a value between $f(a)$ and $f(b)$, then there is a number c in (a, b) such that $f(c) = N$.

- (b) Use the Intermediate Value Theorem to show that there is a point x in the interval $(0, 1)$ at which the graphs of the functions f and g intersect, where

$$f(x) = x^3 + 2x, \quad g(x) = x^2 + 1.$$

Make sure you justify your claims.

Two functions intersect when $f(x) = g(x) \Rightarrow f(x) - g(x) = 0$

Denote $h(x) = f(x) - g(x) = x^3 + 2x - x^2 - 1$

We want to show that $h(x) = 0$ has a solution.

$h(x)$ is continuous everywhere (a polynomial)

and $h(0) = -1 < 0$

$h(1) = 1 > 0$

Therefore, by IVT, there must be a number c in $(0, 1)$ such that

$$h(c) = 0$$

So $f(c) = g(c) \rightarrow f$ and g intersect at a point c in $(0, 1)$.