Homework 3 Solutions

1. (2 pt for process, 1 pt for final answer)

a) $f(x) = x^{87}(5x-1)\sqrt{x}$

Method 1:

$$f(x) = (5x^{88} - x^{87})\sqrt{x}$$

$$f'(x) = (440x^{87} - 87x^{86})\sqrt{x} + \frac{1}{2}x^{-\frac{1}{2}}(5x^{88} - x^{87})$$

Method 2:

$$f'(x) = 87x^{86}(5x-1)\sqrt{x} + 5x^{87}\sqrt{x} + \frac{1}{2}x^{87}(5x-1)x^{-\frac{1}{2}}$$

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b) $g'(x) = \dfrac{(60x^9 + 6x^5)\sqrt{x} - \frac{1}{2}x^{-\frac{1}{2}}(6x^{10} + x^6)}{x}$

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c) $h(t) = \dfrac{t^{50}(2t-4)}{t+3}$

Method 1: $h(t) = \dfrac{(2t^{51} - 4t^{50})}{t+3}$

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$$h'(t) = \frac{(102t^{50} - 200t^{49})(t+3) - (2t^{51} - 4t^{50})}{(t+3)^2}$$

Method 2: $h(t) = \dfrac{t^{50}(2t-4)}{t+3}$

$$h'(t) = \frac{(50t^{49}(2t-4) + 2t^{50})(t+3) - t^{50}(2t-4)}{(t+3)^2}$$

d) $A(y) = 20y^{1/3}\cos y$

$$A'(y) = \frac{20}{3}y^{-2/3}\cos y - 20y^{1/3}\sin y$$

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2. $I(t) = \dfrac{\cos t + 2\sin t + 3}{1 + \frac{t^2}{w}}$ $\quad$ $w > 0$ is a constant

$$I'(t) = \frac{(-\sin t + 2\cos t)\left(1+\frac{t^2}{w}\right) - (\cos t + 2\sin t + 3)\left(\frac{2}{w}t\right)}{\left(1+\frac{t^2}{w}\right)^2}$$ 1pt

$\sin\frac{3\pi}{2} = -1$ $\quad$ $\cos\frac{3\pi}{2} = 0$

$$I'\left(\frac{3\pi}{2}\right) = \frac{(1+2\cdot 0)\left(1+\frac{9\pi^2}{4w}\right) - (0-2+3)\left(\frac{2}{w}\frac{3\pi}{2}\right)}{\left(1+\frac{9\pi^2}{4w}\right)^2} = \frac{1+\frac{9\pi^2}{4w} - \frac{3\pi}{w}}{\left(1+\frac{9\pi^2}{4w}\right)^2}$$ 1pt

this is the m

$$I\left(\frac{3\pi}{2}\right) = \frac{-2+3}{1+\frac{9\pi^2}{4w}} = \frac{1}{1+\frac{9\pi^2}{4w}}$$ ← this is the y 1pt

$y = mx + b$ 1pt

$$\frac{1}{1+\frac{9\pi^2}{4w}} = \frac{1+\frac{9\pi^2}{4w} - \frac{3\pi}{w}}{\left(1+\frac{9\pi^2}{4w}\right)^2} \cdot \frac{3\pi}{2} + b$$

$$b = \frac{1}{1+\frac{9\pi^2}{4w}} - \frac{1+\frac{9\pi^2}{4w} - \frac{3\pi}{w}}{\left(1+\frac{9\pi^2}{4w}\right)^2} \cdot \frac{3\pi}{2}$$ 1pt

$$\boxed{\therefore\ y = \frac{1+\frac{9\pi^2}{4w} - \frac{3\pi}{w}}{\left(1+\frac{9\pi^2}{4w}\right)^2} x + \frac{1}{1+\frac{9\pi^2}{4w}} - \frac{1+\frac{9\pi^2}{4w} - \frac{3\pi}{w}}{\left(1+\frac{9\pi^2}{4w}\right)^2} \cdot \frac{3\pi}{2}}$$ 1pt

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3. $W(u) = \begin{cases} bu+c & u \le \frac{\pi}{2} \\ \cos^2(u)\sin(u) & u > \frac{\pi}{2} \end{cases}$

$$W'(u) = \begin{cases} b & u \le \frac{\pi}{2} \\ 2\cos(u)(-\sin(u))\cdot\sin(u) + \cos^3(u) & u > \frac{\pi}{2} \end{cases}$$

$$= \begin{cases} b & u \le \frac{\pi}{2} \\ -2\sin^2(u)\cos(u) + \cos^3(u) & u > \frac{\pi}{2} \end{cases}$$

2pt

Differentiable means:

- $W(u)$ is continuous
- $W'(u)$ is also continuous

$W(u)$ continuous:

$$b\frac{\pi}{2} + c = \cos^2\left(\frac{\pi}{2}\right)\sin\left(\frac{\pi}{2}\right)$$

$$b\frac{\pi}{2} + c = 0$$

1pt

$W'(u)$ continuous:

$$b = -2\sin^2\left(\frac{\pi}{2}\right)\cos\left(\frac{\pi}{2}\right) + \cos^3\left(\frac{\pi}{2}\right)$$

$$b = 0$$

1pt

$\boxed{\therefore\ b=0,\ c=0}$ 1pt

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4. $f''(x) = -f(x)$

a) $f(x) = \sin(x)$
$f'(x) = \cos x$
$f''(x) = -\sin x$
$f''(x) = -f(x) \Rightarrow -\sin x = -\sin x$ ✓✓ 2pt

b) $f(x) = \cos x$
$f'(x) = -\sin x$
$f''(x) = -\cos x$
$f''(x) = -f(x) \Rightarrow -\cos x = -\cos x$ ✓✓ 2pt

c) $f(x) = \sin x + \cos x$
$f'(x) = \cos x - \sin x$
$f''(x) = -\sin x - \cos x$
$f''(x) = -f(x) \Rightarrow -\sin x - \cos x = -(\sin x + \cos x)$ ✓✓

This means:

$f(x) = A\sin x + B\cos x$
where A and B are arbitrary

or

$f(x) = \sum_{n=0}^{\infty} A_n \sin x + \sum_{m=0}^{\infty} B_m \cos x$
$= A_0 \sin x + A_1 \sin x + \ldots$
$+ B_0 \cos x + B_1 \cos x + \ldots$

2pt

d) say, $f(x) = A\sin x + B\cos x$
$f(0) = A\sin 0 + B\cos 0$
$= B \cdot 1 = B = -1 \Rightarrow B = -1$
$f(\frac{\pi}{2}) = A\sin\frac{\pi}{2} + B\cos\frac{\pi}{2}$
$= A \cdot 1 + 0 = A = 3 \Rightarrow A = 3$

2pt

$$\boxed{\begin{aligned} \therefore f(x) &= A\sin x + B\cos x \\ f(x) &= 3\sin x - \cos x \end{aligned}}$$

Check: $f(x) = 3\sin x - \cos x$
$f'(x) = 3\cos x + \sin x$
$f''(x) = -3\sin x + \cos x$

$f''(x) = -f(x)$ ✓✓

$f(0) = -1$ ✓✓
$f(\frac{\pi}{2}) = 3$ ✓✓

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