MATH 110 Midterm 1, October 24th, 2017

Duration: 90 minutes

*This test has **7 questions** on **8 pages**, for a total of 52 points.*

- Read all the questions carefully before starting to work.
- All questions require a full solution; you should give complete arguments and explanations for all your calculations; answers without justifications will not be marked.
- Continue on the back of the previous page if you run out of space.
- Attempt to answer all questions for partial credit.
- This is a closed-book examination. **None of the following are allowed**: documents, cheat sheets or electronic devices of any kind (including calculators, cell phones, etc.)

First Name: ____________ Last Name: Solution Key

Student-No: ____________ Section: ____________

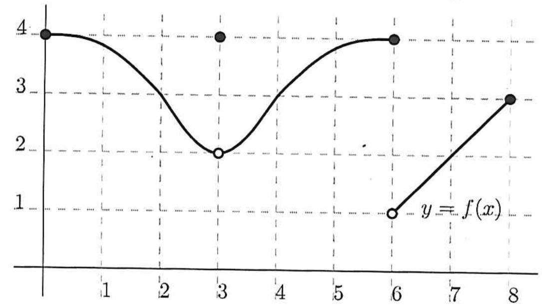
Signature: ____________

Question:	1	2	3	4	5	6	7	Total
Points:	8	6	6	8	8	9	7	52
Score:								

In the following questions, justify your answers and **show all your work.** Unless otherwise indicated, **simplification of answers are required.**

1. This question has two independent problems with multiple parts.

4 marks (a) Consider the function $f(x)$ whose graph is shown below,



Evaluate the following limits or determine they either do not exist or approach (positive or negative) infinity.

(*i*) $\lim_{x\to 0^+} f(x)$ $=4$

(*ii*) $\lim_{x\to 6} f(x)$ $=$ DNE

(*iii*) $\lim_{x\to 3} \dfrac{1}{f(x)-2}$ $= +\infty$ or DNE

(*iv*) $\lim_{h\to 0} \dfrac{f(7+h)-f(7)}{h}$ = slope of f(x) at (x=7) = 1

4 marks (b) Evaluate the following limits or determine they either do not exist or approach (positive or negative) infinity.

(*i*) $\lim_{x\to -2} (x^2+5x+6)$ $=(-2)^2-10+6 = 4-10+6=0$

(*ii*) $\lim_{x\to -2} \dfrac{x^2+5x+6}{x^2+3x+2}$ $= \lim_{x\to -2} \dfrac{\cancel{(x+2)}(x+3)}{(x+1)\cancel{(x+2)}} = \lim_{x\to -2} \dfrac{x+3}{x+1} = \dfrac{1}{-1} = -1$

(*iii*) $\lim_{t\to 2^-} \dfrac{t^3}{8-t^3}$ $= +\infty$

6 marks 2. This question has two independent problems.

(a) (i) Let $f(x) = \dfrac{x}{x+3}$ and $g(x) = \sqrt{1-x}$. Find the domain of $g(f(x))$.

$$g(f(x)) = \sqrt{1 - \frac{x}{x+3}} = \sqrt{\frac{x+3-x}{x+3}} = \sqrt{\frac{3}{x+3}} = \frac{\sqrt{3}}{\sqrt{x+3}}$$

$$x+3 > 0$$

$$x > -3$$

Let (ii) $u(x) = 3x + 2$ and $v(x) = 2x + A$. Find A such that $u(v(x)) = v(u(x))$.

$$u(v(x)) = 3(2x+A)+2 = 6x+3A+2$$

$$v(u(x)) = 2(3x+2)+A = 6x+4+A$$

$$u(v(x)) = v(u(x))$$

$$\cancel{6x}+3A+2 = \cancel{6x}+4+A$$

$$2A = 2$$

$$A = 1$$

6 marks 3. Suppose the temperature $T(t)$ (in degrees °C) in this room at t minutes after the start of the exam has been recorded and reported in the table below.

t	0	5	7	11	15	19	21
$T(t)$	18	18.2	18.5	18.9	19.2	19.2	19.2

(a) Suppose we define the inverse function T^{-1} over the interval $[18, 19]$. Explain the meaning of the output generated by T^{-1}.

T^{-1}: given temperature, find the time

this is the output of T^{-1}

(b) Explain why based on the data available we can claim that the temperature in the room was exactly 19 °C at some point in time.

$T(t)$ is a continuous function, $T(11) = 18.9$°C and $T(15) = 19.2$°C, thus by IVT the temperature must be exactly 19°C at some point in time

(c) Estimate $T'(19)$. Provide a rationale for your estimate.

use average change to estimate

Correct answer 1:	Correct answer 2:
$t=15$ and $t=19$:	$t=19$ and $t=21$
$\frac{19.2-19.2}{19-15} = 0$	$\frac{19.2-19.2}{21-19} = 0$

both methods yield the same answer

(d) Estimate at what rate is the temperature in the room changing at $t = 7$.

again, use average change to estimate

Correct answer 1:	Correct answer 2:
$t=5$ and $t=7$:	$t=7$ and $t=11$:
$\frac{18.5-18.2}{7-5} = \frac{0.3}{2} = 0.15 \frac{°C}{min}$	$\frac{18.9-18.5}{11-7} = \frac{0.4}{4} = 0.1 \frac{°C}{min}$

both are fine

8 marks 4. (a) Find all values of x where $H(x)$ is continuous, where

Continuous: $\lim_{x \to a} f(x) = f(a)$

$$H(x) = \begin{cases} \dfrac{x^2+5x+6}{x+3}, & x \le 0 \quad \Leftarrow x \ne -3 \\ \dfrac{x^2}{x^3+x^2}, & x > 0 \quad \Leftarrow \begin{array}{l} x^3+x^2 \ne 0 \\ x^2(x+1) \ne 0 \end{array} \end{cases}$$

$x \ne 0$, $x \ne -1$ both are not with domain, however

Boundary $x=0$:

$H(0) = 2$

$\lim_{x \to 0^-} H(x) = 2$

$\lim_{x \to 0^+} H(x) = \lim_{x \to 0^+} \frac{1}{x+1} = 1$ } not continuous at $x=0$

$\therefore$ not continuous at $x=0, x=3$

interval of continuity:

$x \in (-\infty, 0) \cup (0,3) \cup (3, \infty)$

(b) Find a and b such that the following function is continuous everywhere

$$g(y) = \begin{cases} \sqrt{3-y}, & y < -1 \\ ay+b, & -1 \le y \le 1 \\ 10y-8, & y > 1 \end{cases}$$

$y=-1$:

$\sqrt{3-(-1)} = -a+b$

$2 = b-a$

$y=1$:

$a+b=2$

$b-a=2$
$b+a=2$

$2b=4 \quad b=2, a=0$

*** You should use limits ***

(c) Find a valid function $L(t)$ such that the following function $F(t)$ is continuous everywhere:

$$F(t) = \begin{cases} 3^t, & t < 0 \\ L(t), & 0 \le t \le 5 \\ \frac{1}{t+3}, & t > 5 \end{cases}$$

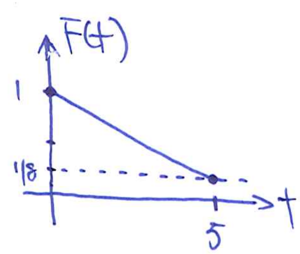
$t=0$:

$3^0 = L(0)$

$t=5$:

$L(5) = \frac{1}{8}$

$L(0) = 1$
$L(5) = \frac{1}{8}$



slope: $\dfrac{\frac{1}{8}-1}{5-0} = \dfrac{-7/8}{5} = -\dfrac{7}{40}$

y-intercept $= 1$

$\therefore L(t) = -\frac{7}{40}t + 1$

*** You should use limits ***

*** $L(t)$ can be any function as long as $L(0)=1$ and $L(5)=\frac{1}{8}$; in this case, I used a straight line, but you can be creative ***

8 marks 5. Consider the function $f(t) = 3t^2 + t$ for $t \geq 0$.

Let $P(t_P, y_P)$ and $Q(t_Q, y_Q)$ be two points on the graph of f with $t_P = 2$ and $t_Q = 2 + h$, for $h \neq 0$.

(a) Find the slope m_{PQ} of the line that goes through P and Q. Simplify your answer.

$t_P = 2,\ y_P = 3\cdot 2^2 + 2 = 14 \Rightarrow P = (2, 14)$

$t_Q = 2+h,\ y_Q = 3(2+h)^2 + 2 + h = 3(4+4h+h^2) + 2 + h = 3h^2 + 13h + 14 \Rightarrow Q = (2+h, 3h^2+13h+14)$

$$m_{PQ} = \frac{3h^2+13h+14-14}{2+h-2} = \frac{3h^2+13h}{h} = 3h+13$$

(b) Compute $\lim_{h \to 0} m_{PQ}$ and explain what this limit represents graphically.

$$\lim_{h\to 0} m_{PQ} = \lim_{h \to 0}(3h+13) = 13$$

this is the instantaneous rate of change (or slope of tangent line) at $t=2$

(c) Suppose $f(t)$ represents the position (in m) of a car at time t (in s).

(*i*) If $h = 1$, what does m_{PQ} represent in this case?

average rate of change (average velocity) between $t=2$ and $t=3$

(*ii*) What is the instantaneous velocity of the car at $t = 2$?

13 m/s (as in (b))

(*iii*) At $t = 2$, is the car moving away from or towards its initial position? Explain.

away, velocity is positive

9 marks

6. This question has three independent problems. You may use differentiation rules to compute derivatives as appropriate.

(a) Find y' if $y = 2x^5 + \sqrt[3]{x^2} - \dfrac{3}{\sqrt{x^3}} + 120^{-1/5}$.

$$y = 2x^5 + x^{2/3} - 3x^{-3/2} + 120^{-\frac{1}{5}}$$

$$y' = 10x + \frac{2}{3}x^{-\frac{1}{3}} + \frac{9}{2}x^{-5/2}$$

(b) For what value(s) of x does the graph of $f(x) = x^3 + 3x^2 + x + 3$ have a horizontal tangent line?

horizontal tangent line $= (\text{slope} = 0) = f'(x) = 0$

$$f'(x) = 3x^2 + 6x + 1 = 0$$

$$x = \frac{-6 \pm \sqrt{36-12}}{6} = \frac{-6 \pm \sqrt{24}}{6} = \frac{-6 \pm 2\sqrt{6}}{6} = -1 \pm \frac{1}{3}\sqrt{6}$$

(c) Find an equation of the line tangent to the graph of $y = 1 + 4x + x^2$ at the point of coordinate $x = 1$.

$y' = 4 + 2x$

at $x = 1$:

$y(1) = 1 + 4 + 1 = 6$

$y'(1) = 6$

$y = mx + b$

$y = 6x + b$

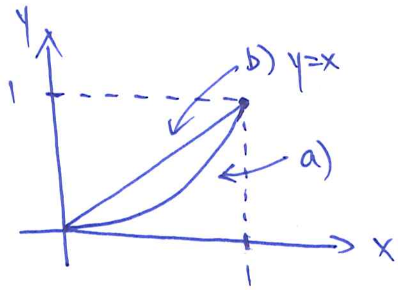
at $(1, 6)$:

$6 = 6 \cdot 1 + b \quad b = 0$

$\therefore y = 6x$

7 marks

7. (a) Sketch the graph of a continuous function f defined on $[0, 1]$ and with range $[0, 1]$. Make sure you label the coordinate axes. Your sketch must be to scale.



*** Any graph is correct as long as domain and range are both $[0,1]$ ***

(b) Draw the line $y = x$ on the same diagram you drew in part (a). Based on your sketch, how many intersections does your function f have with the line $y = x$, that is, how many solutions does $f(x) = x$ have?

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*** again, depending on the function you draw, you may have different number of solutions ***

(c) Using appropriate theorem(s) discussed in class, prove that *any* function f defined on $[0, 1]$ and with range $[0, 1]$ must have at least one point x in its domain such that $f(x) = x$. (Hint: Consider the function $F(x) = f(x) - x$.)

for any $f(x)$ defined in $[0,1]$ with range $[0,1]$, let $F(x)=f(x)-x$

- $F(0)=f(0)-0=f(0)$
- $F(1)=f(1)-1$

since $f(x)$ has a range of $[0,1]$, We know that $0 \le f(0) \le 1$ and $0 \le f(1) \le 1$

thus $\underline{F(0)=f(0) \ge 0}$, and $\underline{F(1)=f(1)-1 \le 0}$

- If $F(0)=f(0)>0$ and $F(1)=f(1)-1<0$, then there is a sign change, since $F(x)=f(x)-x$ is continuous, by IVT there must be solution to $f(x)=x$
- If $F(0)=f(0)=0$ or $F(1)=f(1)-1=0$, this means a solution is at either the start point $x=0$ or at the endpoint $x=1$, therefore $f(x)=x$ has solution(s)

EITHER CASE, $f(x)=x$ has solution(s)

*** this is a difficult question ***