

MATH 110 Midterm 1, October 24th, 2017

Duration: 90 minutes

This test has 7 questions on 8 pages, for a total of 52 points.

- Read all the questions carefully before starting to work.
- All questions require a full solution; you should give complete arguments and explanations for all your calculations; answers without justifications will not be marked.
- Continue on the back of the previous page if you run out of space.
- Attempt to answer all questions for partial credit.
- This is a closed-book examination. **None of the following are allowed:** documents, cheat sheets or electronic devices of any kind (including calculators, cell phones, etc.)

First Name: _____ Last Name: Solution Key

Student-No: _____ Section: _____

Signature: _____

Question:	1	2	3	4	5	6	7	Total
Points:	8	6	6	8	8	9	7	52
Score:								

Student Conduct during Examinations

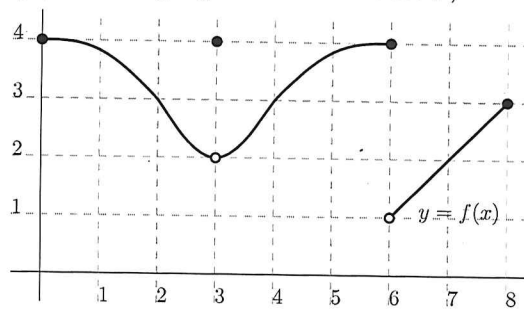
1. Each examination candidate must be prepared to produce, upon the request of the invigilator or examiner, his or her UBCcard for identification.
2. Examination candidates are not permitted to ask questions of the examiners or invigilators, except in cases of supposed errors or ambiguities in examination questions, illegible or missing material, or the like.
3. No examination candidate shall be permitted to enter the examination room after the expiration of one-half hour from the scheduled starting time, or to leave during the first half hour of the examination. Should the examination run forty-five (45) minutes or less, no examination candidate shall be permitted to enter the examination room once the examination has begun.
4. Examination candidates must conduct themselves honestly and in accordance with established rules for a given examination, which will be articulated by the examiner or invigilator prior to the examination commencing. Should dishonest behaviour be observed by the examiner(s) or invigilator(s), pleas of accident or forgetfulness shall not be received.
5. Examination candidates suspected of any of the following, or any other similar practices, may be immediately dismissed from the examination by the examiner/invigilator, and may be subject to disciplinary action:
 - (i) speaking or communicating with other examination candidates, unless otherwise authorized;
 - (ii) purposely exposing written papers to the view of other examination candidates or imaging devices;
 - (iii) purposely viewing the written papers of other examination candidates;
 - (iv) using or having visible at the place of writing any books, papers or other memory aid devices other than those authorized by the examiner(s); and,
 - (v) using or operating electronic devices including but not limited to telephones, calculators, computers, or similar devices other than those authorized by the examiner(s)(electronic devices other than those authorized by the examiner(s) must be completely powered down if present at the place of writing).
6. Examination candidates must not destroy or damage any examination material, must hand in all examination papers, and must not take any examination material from the examination room without permission of the examiner or invigilator.
7. Notwithstanding the above, for any mode of examination that does not fall into the traditional, paper-based method, examination candidates shall adhere to any special rules for conduct as established and articulated by the examiner.
8. Examination candidates must follow any additional examination rules or directions communicated by the examiner(s) or invigilator(s).

In the following questions, justify your answers and **show all your work**. Unless otherwise indicated, **simplification of answers are required**.

1. This question has two independent problems with multiple parts.

4 marks

- (a) Consider the function $f(x)$ whose graph is shown below,



Evaluate the following limits or determine they either do not exist or approach (positive or negative) infinity.

$$(i) \lim_{x \rightarrow 0^+} f(x) = 4$$

$$(ii) \lim_{x \rightarrow 6} f(x) = \text{DNE}$$

$$(iii) \lim_{x \rightarrow 3} \frac{1}{f(x) - 2} = +\infty \text{ or } \text{DNE}$$

$$(iv) \lim_{h \rightarrow 0} \frac{f(7+h) - f(7)}{h} = \text{slope of } f(x) \text{ at } (x=7) = 1$$

4 marks

- (b) Evaluate the following limits or determine they either do not exist or approach (positive or negative) infinity.

$$(i) \lim_{x \rightarrow -2} (x^2 + 5x + 6) = (-2)^2 - 10 + 6 = 4 - 10 + 6 = 0$$

$$(ii) \lim_{x \rightarrow -2} \frac{x^2 + 5x + 6}{x^2 + 3x + 2} = \lim_{x \rightarrow -2} \frac{(x+2)(x+3)}{(x+1)(x+2)} = \lim_{x \rightarrow -2} \frac{x+3}{x+1} = \frac{1}{-1} = -1$$

$$(iii) \lim_{t \rightarrow 2^-} \frac{t^3}{8 - t^3} = +\infty$$

6 marks 2. This question has two independent problems.

(a) (i) Let $f(x) = \frac{x}{x+3}$ and $g(x) = \sqrt{1-x}$. Find the domain of $g(f(x))$.

$$g(f(x)) = \sqrt{1 - \frac{x}{x+3}} = \sqrt{\frac{x+3-x}{x+3}} = \sqrt{\frac{3}{x+3}} = \frac{\sqrt{3}}{\sqrt{x+3}}$$

$$x+3 > 0$$

$$x > -3$$

Let (ii) $u(x) = 3x + 2$ and $v(x) = 2x + A$. Find A such that $u(v(x)) = v(u(x))$.

$$u(v(x)) = 3(2x+A) + 2 = 6x + 3A + 2$$

$$v(u(x)) = 2(3x+2) + A = 6x + 4 + A$$

$$u(v(x)) = v(u(x))$$

$$\cancel{6x} + 3A + 2 = \cancel{6x} + 4 + A$$

$$2A = 2$$

$$A = 1$$

- 6 marks 3. Suppose the temperature $T(t)$ (in degrees $^{\circ}\text{C}$) in this room at t minutes after the start of the exam has been recorded and reported in the table below.

t	0	5	7	11	15	19	21
$T(t)$	18	18.2	18.5	18.9	19.2	19.2	19.2

- (a) Suppose we define the inverse function T^{-1} over the interval $[18, 19]$. Explain the meaning of the output generated by T^{-1} .

T^{-1} : given temperature, find the time
 ↑ this is the output of T^{-1}

- (b) Explain why based on the data available we can claim that the temperature in the room was exactly 19°C at some point in time.

$T(t)$ is a continuous function, $T(11) = 18.9^{\circ}\text{C}$ and $T(15) = 19.2^{\circ}\text{C}$, thus by IVT the temperature must be exactly 19°C at some point in time

- (c) Estimate $T'(19)$. Provide a rationale for your estimate.

use average change to estimate

Correct answer 1: $t=15$ and $t=19$: $\frac{19.2-19.2}{19-15} = 0$	Correct answer 2: $t=19$ and $t=21$: $\frac{19.2-19.2}{21-19} = 0$
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} both methods yield the same answer

- (d) Estimate at what rate is the temperature in the room changing at $t = 7$.

again, use average change to estimate

Correct answer 1: $t=5$ and $t=7$: $\frac{18.5-18.2}{7-5} = \frac{0.3}{2} = 0.15 \frac{^{\circ}\text{C}}{\text{min}}$	Correct answer 2: $t=7$ and $t=11$: $\frac{18.9-18.5}{11-7} = \frac{0.4}{4} = 0.1 \frac{^{\circ}\text{C}}{\text{min}}$
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both are fine

8 marks

4. (a) Find all values of x where $H(x)$ is continuous, where

Continuous:

$$\lim_{x \rightarrow a} f(x) = f(a)$$

$$H(x) = \begin{cases} \frac{x^2 + 5x + 6}{x + 3}, & x \leq 0 \quad \Leftarrow x \neq -3 \\ \frac{x^2}{x^3 + x^2}, & x > 0 \quad \Leftarrow \begin{matrix} x^3 + x^2 \neq 0 \\ x^2(x+1) \neq 0 \end{matrix} \end{cases}$$

$x \neq 0, x \neq -1$
both are not with domain, however

Boundary $x=0$:

$$H(0) = 2$$

$$\lim_{x \rightarrow 0^-} H(x) = 2$$

$$\lim_{x \rightarrow 0^+} H(x) = \lim_{x \rightarrow 0^+} \frac{1}{x+1} = 1$$

} not continuous at $x=0$

\therefore not continuous at $x=0, x=3$

interval of continuity:

$$x \in (-\infty, 0) \cup (0, 3) \cup (3, \infty)$$

(b) Find a and b such that the following function is continuous everywhere

$$g(y) = \begin{cases} \sqrt{3-y}, & y < -1 \\ ay + b, & -1 \leq y \leq 1 \\ 10y - 8, & y > 1 \end{cases}$$

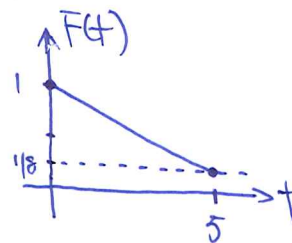
$$\left. \begin{array}{l} y = -1: \\ \sqrt{3-(-1)} = -a + b \\ 2 = b - a \\ \\ y = 1: \\ a + b = 2 \end{array} \right\} \begin{array}{l} b - a = 2 \\ b + a = 2 \\ \hline 2b = 4 \quad b = 2, a = 0 \end{array}$$

*** You should use limits ***

(c) Find a valid function $L(t)$ such that the following function $F(t)$ is continuous everywhere:

$$F(t) = \begin{cases} 3^t, & t < 0 \\ L(t), & 0 \leq t \leq 5 \\ \frac{1}{t+3}, & t > 5 \end{cases}$$

$$\left. \begin{array}{l} t = 0: \\ 3^0 = L(0) \\ \\ t = 5: \\ L(5) = \frac{1}{8} \end{array} \right\} \begin{array}{l} L(0) = 1 \\ L(5) = \frac{1}{8} \end{array}$$



$$\text{slope} = \frac{\frac{1}{8} - 1}{5 - 0} = \frac{-7/8}{5} = -\frac{7}{40}$$

y-intercept = 1

$$\therefore L(t) = -\frac{7}{40}t + 1$$

*** You should use limits ***

*** $L(t)$ can be any function as long as $L(0) = 1$ and $L(5) = \frac{1}{8}$; in this case, I used a straight line, but you can be creative ***

8 marks

5. Consider the function $f(t) = 3t^2 + t$ for $t \geq 0$.

Let $P(t_P, y_P)$ and $Q(t_Q, y_Q)$ be two points on the graph of f with $t_P = 2$ and $t_Q = 2 + h$, for $h \neq 0$.

(a) Find the slope m_{PQ} of the line that goes through P and Q . Simplify your answer.

$$t_P = 2, y_P = 3 \cdot 2^2 + 2 = 14 \Rightarrow P = (2, 14)$$

$$t_Q = 2+h, y_Q = 3(2+h)^2 + 2+h = 3(4+4h+h^2) + 2+h = 3h^2 + 13h + 14 \Rightarrow Q = (2+h, 3h^2 + 13h + 14)$$

$$m_{PQ} = \frac{3h^2 + 13h + 14 - 14}{2+h-2} = \frac{3h^2 + 13h}{h} = 3h + 13$$

(b) Compute $\lim_{h \rightarrow 0} m_{PQ}$ and explain what this limit represents graphically.

$$\lim_{h \rightarrow 0} m_{PQ} = \lim_{h \rightarrow 0} (3h + 13) = 13$$

this is the instantaneous rate of change (or slope of tangent line) at $t=2$

(c) Suppose $f(t)$ represents the position (in m) of a car at time t (in s).

(i) If $h = 1$, what does m_{PQ} represent in this case?

average rate of change (average velocity) between $t=2$ and $t=3$

(ii) What is the instantaneous velocity of the car at $t = 2$?

13 m/s (as in (b))

(iii) At $t = 2$, is the car moving away from or towards its initial position? Explain.

away, velocity is positive

- 9 marks 6. This question has three independent problems. You may use differentiation rules to compute derivatives as appropriate.

(a) Find y' if $y = 2x^5 + \sqrt[3]{x^2} - \frac{3}{\sqrt{x^3}} + 120^{-1/5}$.

$$y = 2x^5 + x^{2/3} - 3x^{-3/2} + 120^{-1/5}$$

$$y' = 10x + \frac{2}{3}x^{-1/3} + \frac{9}{2}x^{-5/2}$$

(b) For what value(s) of x does the graph of $f(x) = x^3 + 3x^2 + x + 3$ have a horizontal tangent line?

$$\text{horizontal tangent line} = (\text{slope} = 0) = f'(x) = 0$$

$$f'(x) = 3x^2 + 6x + 1 = 0$$

$$x = \frac{-6 \pm \sqrt{36 - 12}}{6} = \frac{-6 \pm \sqrt{24}}{6} = \frac{-6 \pm 2\sqrt{6}}{6} = -1 \pm \frac{1}{3}\sqrt{6}$$

(c) Find an equation of the line tangent to the graph of $y = 1 + 4x + x^2$ at the point of coordinate $x = 1$.

$$y' = 4 + 2x$$

$$\text{at } x = 1 :$$

$$y(1) = 1 + 4 + 1 = 6$$

$$y'(1) = 6$$

$$y = mx + b$$

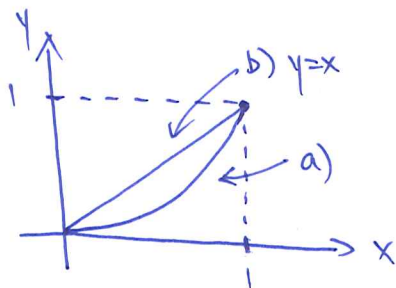
$$y = 6x + b$$

$$\text{at } (1, 6) :$$

$$6 = 6 \cdot 1 + b \quad b = 0$$

$$\therefore y = 6x$$

7. (a) Sketch the graph of a continuous function f defined on $[0, 1]$ and with range $[0, 1]$. Make sure you label the coordinate axes. Your sketch must be to scale.



*** Any graph is correct as long as domain and range are both $[0, 1]$ ***

- (b) Draw the line $y = x$ on the same diagram you drew in part (a). Based on your sketch, how many intersections does your function f have with the line $y = x$, that is, how many solutions does $f(x) = x$ have?

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*** again, depending on the function you draw, you may have different number of solutions ***

- (c) Using appropriate theorem(s) discussed in class, prove that *any* function f defined on $[0, 1]$ and with range $[0, 1]$ must have at least one point x in its domain such that $f(x) = x$. (Hint: Consider the function $F(x) = f(x) - x$.)

for any $f(x)$ defined in $[0, 1]$ with range $[0, 1]$, let
 $F(x) = f(x) - x$

$$\begin{aligned} \square F(0) &= f(0) - 0 = f(0) \\ \square F(1) &= f(1) - 1 \end{aligned} \left. \begin{array}{l} \text{since } f(x) \text{ has a range of } [0, 1], \\ \text{we know that } 0 \leq f(0) \leq 1 \text{ and} \\ 0 \leq f(1) \leq 1 \end{array} \right\}$$

thus $\underline{F(0) = f(0) \geq 0}$, and $\underline{F(1) = f(1) - 1 \leq 0}$

- If $F(0) = f(0) > 0$ and $F(1) = f(1) - 1 < 0$, then there is a sign change, since $F(x) = f(x) - x$ is continuous, by IVT there must be solution to $f(x) = x$
- If $F(0) = f(0) = 0$ or $F(1) = f(1) - 1 = 0$, this means a solution is at either the start point $x=0$ or at the endpoint $x=1$, therefore $f(x) = x$ has solution(s)

EITHER CASE, $f(x) = x$ has solution(s)

*** this is a difficult question ***