Video 19 - Deriving the Ex and Cum Dividend Formulas

The following is a supplementary transcript for tutorial videos from

https://blogs.ubc.ca/financefundamentals/

Welcome back! This video will be a continuation of the video where we learned about the 3 main approaches to calculating the price of a stock. We have the perpetuity formula that allows us to calculate the price of a stock with constant dividends. There is the Gordon Growth Model that allows us to calculate the price of a stock with constantly growing dividends. And lastly, the Multistage Growth Model is applied to calculate the price of a stock that experiences dividends that grow at different stages.

Video at 00:39

Specifically in regards to the Gordon Growth Model, there are two different versions of the Gordon Growth Model that can be used in different scenarios: the ex dividend formula and the cum dividend formula. In this video, we will be exploring how to derive the ex dividend formula, how to derive the cum divided formula, and how the ex dividend formula and the cum dividend formula are essentially two sides of the same concept. As a warning, this video will be quite math-heavy; however, the purpose of exploring the derivations of these formulas is to help you better understand where these formulas came from. Throughout this video, we will also be using multiple examples to better illustrate why these formulas work, and how they can be used. So buckle up, and let's get this show on the road!

Video at 01:33

Recall that the Gordon Growth Model allows us to calculate the price of a stock that raises the size of each consecutive dividend at a constant rate, and there are two versions of the Gordon Growth Model: the ex dividend formula and the cum dividend formula.

$$price = PV_{ex} = \frac{D_1}{r_E - g}$$

$$price = PV_{cum} = \frac{D_0 \times (1+r)}{r_E - g}$$

In both formulas, "g" is the constant growth rate of the dividends, and r_E is the cost of the company's equity, which is the appropriate discount rate that reflects the riskiness of the firm's equity relative to the rest of the stock market.

Video at 02:03

In the ex dividend formula, the numerator is D_1, the amount of next period's dividend. Thus, the ex dividend formula is used to calculate the price of a stock in situations where we, as an investor, just missed a dividend payment and anticipate on receiving the next dividend, which is in exactly one period. On the other hand, the cum dividend formula has D_0 in the numerator, which is the amount of this period's dividend; thus, the cum dividend formula is used to calculate the price of a stock in situations where this period's dividend is just about to be paid. In other words, as an investor, we anticipate that the dividend is just about to *come*.

Video at 02:46

But why are these formulas the way they are? Let's start off with the derivation of the ex dividend formula. If we were to draw the dividend cash flows on a timeline, we are trying to value the price or present value of a stock in today's dollars, which is at time t=0, by forecasting the future expected dividends from the stock and discounting them all the way back to today. For illustration purposes, let's imagine that a company pays annual dividends, and the dividend today (D_0) was just paid; thus, when we are forecasting the future expected dividends, the first dividend that we are concerned with is the one that will be paid out one period (or one year) from today, which is D_1. We can basically expect dividends every year until the end of time. Additionally, with the Gordon Growth Model, these dividends will grow at a constant rate "g".

Video at 03:41

Note that many students confuse a constant rate with constant dividends. Constant dividends would suggest that dividends are the same amount each period; however, a constant rate suggests that the dividends are growing, but at the same rate each period. To illustrate, a constant rate "g" tells us that $D_2 = D_1 \times (1+g)$, and $D_3 = D_2 \times (1+g) \ = \ D_1 \times (1+g)^2$, and so on, as all of the dividends can be expressed in terms of D_1 (Figure 1).

Video at 04:18

Since the stock price reflects the present value of all the dividends, these future dividends are discounted back to today at the cost of equity, r_E. From here, we can see a pattern: each dividend term is multiplied by (1+g) to the power of one less the time period in the numerator, and (1+r) to the power of the time period in the denominator: $\frac{(1+g)^{t-1}}{(1+r)^t}$.

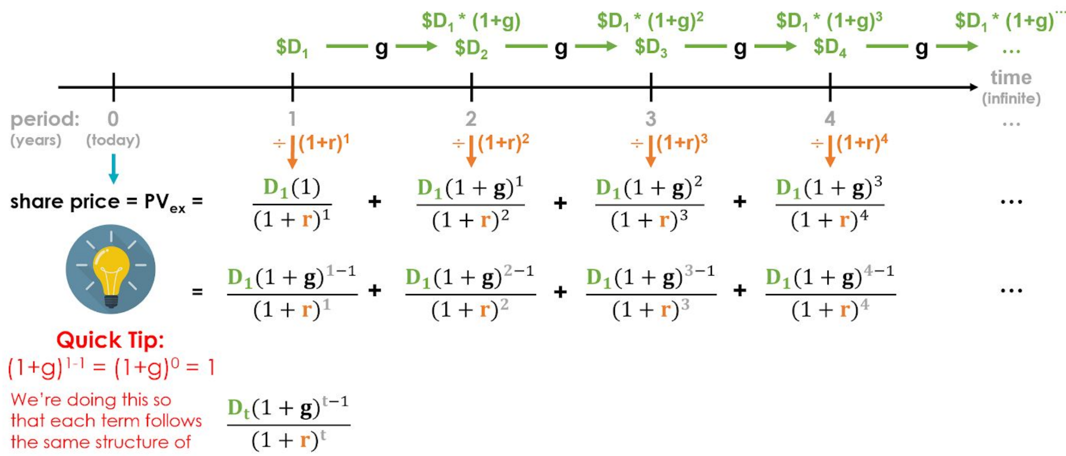


Figure 1

Video at 04:47

If we factor out D_1 from each dividend term, we can more clearly see this sequential pattern. Furthermore, we can factor out (1+r) from the sequence, so that the exponents in the numerator and the denominator of each term now matches (ie: we factored out $\frac{D_1}{1+r}$ so that each dividend term is multiplied by $\frac{(1+g)^{t-1}}{(1+r)^{t-1}}$).

Video at 05:17

From here, we have a pattern in the future expected dividends: what we have factored out is the present value of the first dividend ($\frac{D_1}{1+r}$), and in the brackets, we can see that the present value of each future dividend becomes increasingly larger by $\frac{1+g}{1+r}$. In other words, the present value of $D_2 = PV(D_1) \times \frac{1+g}{1+r} = \frac{D_1}{1+r} \times \frac{1+g}{1+r}$ and so on, for infinitely many dividends.

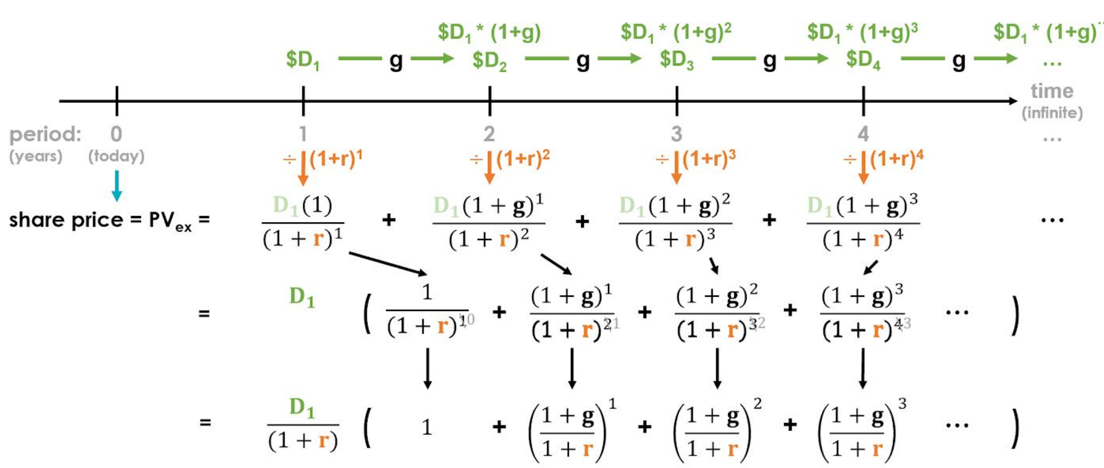


Figure 2

Video at 05:45

So, how can we find the sum of these dividends that keep growing forever? Thinking back to high school, this pattern, which is an infinite geometric sequence, may look familiar to you. Let's take a trip down memory lane together. As a quick recap, a geometric sequence is a progression of numbers, where the next number in the sequence is found by multiplying the previous number by a fixed rate, known as the common ratio. For example, we can build a series by starting with the number 1 and multiplying every term from here on out by half. Sure enough, our series will start growing from 1 to $\frac{1}{2}$ to $\frac{1}{4}$ to $\frac{1}{8}$ to $\frac{1}{16}$ to $\frac{1}{32}$ and so on. But if this sequence can continue for infinitely many terms, as the next term keeps getting multiplied by half, how do we find the sum of this geometric sequence?

Video at 06:38

Well eventually, the terms will start getting so infinitesimally small that the contribution of the next term to the overall sum of the series is pretty much negligible. Thus, we can calculate the sum of an infinite geometric series with the following formula:

$$\lim_{n \to \infty} S_n = \frac{a}{1-x}$$

where "a" is the first term in the series, and “x” is the common ratio, which is the constant rate that the terms in the sequence grows by.

Video at 07:06

The sum can only be found if the absolute value of the common ratio is less than 1: |x|<1. Within this range, each term in the series will get smaller and smaller until they approach zero, at which point the sum will converge to a definite number. You can imagine that if the common ratio was, say, equal to 2, then the series will keep growing indefinitely (ie: 2 to 4 to 8 to 16, etc...). Thus, the sum would be infinite. If the concept of geometric series is fuzzy in your memory, there is an abundance of resources online, including walkthroughs of the derivation of this formula on YouTube. But for now, let's run with this formula so that we can finish what we came here to do, which is to derive the ex dividend and cum dividend formulas.

Video at 07:47

Where we left off with the derivation of the ex dividend formula, we realized that the stream of future dividends follows a pattern, where the first term is the present value of the first dividend, and each term is equal to the previous term multiplied by $\frac{1+g}{1+r}$, which we now know is called the common ratio. Of course, this is the same structure as an infinite geometric series.

Video at 08:10

To make this more evident, we can substitute the common ratio, $\frac{1+g}{1+r}$, with a variable, such as "x" (Figure 3). Following the formula for the sum of an infinite geometric series, we know that:

$$price \;=\; sum\ of\ PVs\ of\ all\ future\ dividends = \lim_{n\to\infty} S_n = \frac{\frac{D_1}{1+r}}{1-x}$$

Video at 08:36

From here on, we will perform some simple algebra to simplify the equation (Figure 3). We can substitute our common ratio of $\frac{1+g}{1+r}$ back in for "x". We are now left with this complex fraction, which we can condense by rewriting 1 (from “1-x” in the denominator of the sum) as $\frac{1+r}{1+r}$ so that the denominator is now expressed as one fraction. This complex fraction can be rewritten as a regular fraction by taking the reciprocal, leaving us with (1+r) in the numerator and (r-g) in the denominator ($\frac{1+r}{r-g}$). Finally, the (1+r) in both the numerator and denominator cancel out, leaving us with a much friendlier equation:

$$\text{price} = \lim_{n\to\infty} S_n = \frac{D_1}{r_E - g}$$

The price of a stock, which is the present value of the infinitely many future dividends, is equal to D_1 (next year's dividend), divided by r_E (the cost of equity) minus "g" (the constant growth rate of the dividends).

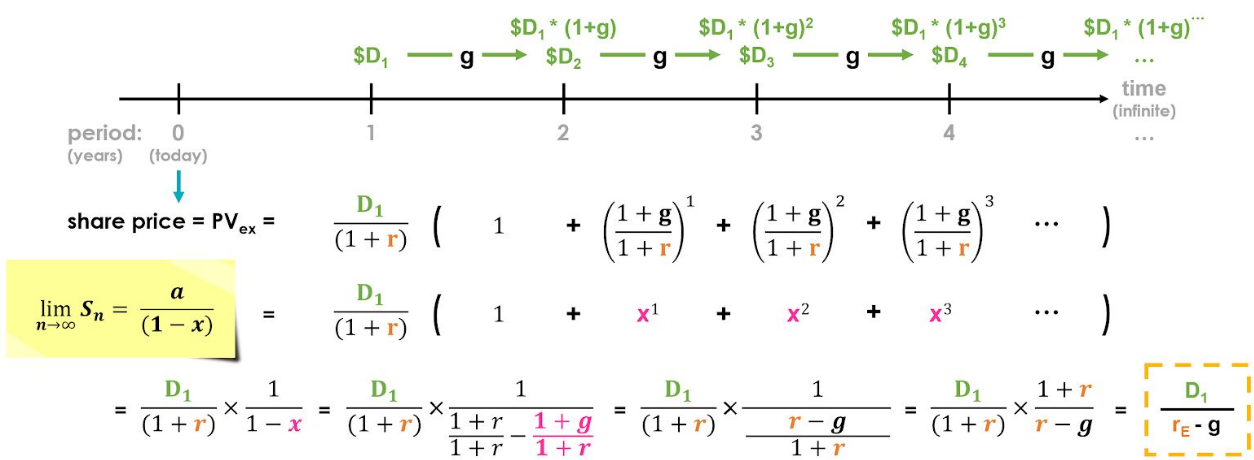


Figure 3

Video at 09:34

Recall, that the sum of an infinite geometric series could only be calculated if the absolute value of the common ratio is less than 1: |x|<1, allowing the numbers in the sequence to get smaller so that the sum would also approach a finite number. When we take a look at the ex dividend formula, where the common ratio is equal to $\frac{1+g}{1+r}$, we want to ensure that the numerator is smaller than the denominator, so that this fraction is less than 1. This will always be true if $r_E > g$. In other words, the discount rate must be greater than the growth rate for us to be able to calculate a finite value for the price of the stock.

Video at 10:13

Intuitively, this should make sense if we think about our cash flows. Imagine, instead, that "g" is larger than r_E. What this means is that, even after discounting the dividends, the present value of each dividend is growing even more, and as the dividends get larger and larger forever, the present value would be infinite.

Video at 10:35

In reality, it is possible for dividends to experience supernormal growth, such as in tech startups like Uber and Tesla, in which case, we can use the Multistage Growth Model that we learned about in the previous video ("Stock Valuation: Multistage Growth Model"). But no firm can sustain a supernormal growth rate forever, which is why we don't have to worry too much about ensuring that r_E is greater than "g" to use the ex dividend formula.

Video at 11:01

Finally, with the ex dividend formula, sometimes we are given the value for D_0, the dividend that was just paid, such as in the example with Michael and Nike in the previous video. Take a moment here to pause the video to refresh yourself.

Recall: The current market price for Nike is $81. Michael just missed the annual dividend of $2/share that was paid this morning. Based on Nike's financial data, the effective annual discount rate for Nike's equity is 11%, and its dividends will grow at a rate of 8%. Should Michael invest in Nike?

Video at 11:18

With this example, we applied the ex dividend formula, but we have to calculate D_1, the numerator, for ourselves. This is not a problem, as we know that the dividends grow at a constant rate of "g", which, in this case, was equal to 8%. Thus, we found that

$$D_1 = D_0 \times (1+g) = \$2 \times (1.08) = \$2.16$$

Generally speaking, we can modify the ex dividend formula and express the numerator as

$$price = PV_{ex} = \frac{D_1}{r_E - g} = \frac{D_0 \times (1+g)}{r_E - g}$$

for the scenarios where we are given the value of the dividend today (D_0) that we just missed out on, instead of the value of the dividend in one period from now that we anticipate on receiving next (D_1).

Video at 12:02

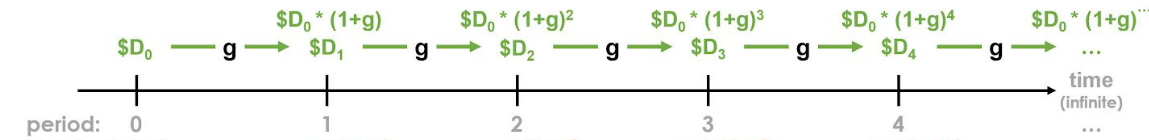
Now that we have walked through the derivation of the ex dividend formula, we can quickly understand how the cum dividend formula is derived. We already know that the difference between the ex dividend and the cum dividend formula is that, when we use the cum dividend formula, we are expecting to receive D_0, the dividend that will be paid this period, which is missed out on in the scenarios where we use the ex dividend formula. Thus, the cum dividend formula not only considers the infinitely many future dividends, but must also take into account D_0.

Video at 12:33

Following the same steps that we used to derive the ex dividend formula, we first rewrite the dividends to be expressed in terms of our first term, D_0 (Figure 4). Note that with the ex dividend formula, our first term was D_1, which is why the ex dividend formula was based on D_1. But, as you will soon notice with the cum dividend formula, we will now be basing all the cash flows on D_0, the dividend we anticipate on receiving this period.

Video at 12:58

Just like before, since the stock price reflects the present value of all the dividends, these future dividends are discounted back to today at the cost of equity, r_E. From here, we can see a pattern: each dividend term is multiplied by $\left(\frac{1+g}{1+r}\right)^t$. If we factor out D_0, we can more clearly see this sequential pattern. Again, we can see a structure that is similar to that of the sum of an infinite geometric series, where D_0 is the first term, as it is the first dividend that we expect to receive, and each term is equal to the previous term multiplied by $\frac{1+g}{1+r}$, our common ratio.



$$D_0 + \frac{D_0(1+g)^1}{(1+r)^1} + \frac{D_0(1+g)^2}{(1+r)^2} + \frac{D_0(1+g)^3}{(1+r)^3} + \frac{D_0(1+g)^4}{(1+r)^4} \cdots$$

$$= D_0\left(1 + \left(\frac{1+g}{1+r}\right)^1 + \left(\frac{1+g}{1+r}\right)^2 + \left(\frac{1+g}{1+r}\right)^3 + \left(\frac{1+g}{1+r}\right)^4 \cdots\right)$$

$$\left(\frac{1+g}{1+r}\right)^1 \quad \left(\frac{1+g}{1+r}\right)^1 \quad \left(\frac{1+g}{1+r}\right)^1$$

Figure 4

Video at 13:48

To make the structure easier to follow, we can substitute the common ratio $\frac{1+g}{1+r}$ with a variable, such as "x" (Figure 5). Thus, following the formula for the sum of an infinite geometric series, we know that:

$$price = sum\ of\ PVs\ of\ all\ future\ dividends = \lim_{n\to\infty} S_n = \frac{D_0}{1-x}$$

Video at 14:09

From here on, we will perform the same simple algebra to simplify the equation (Figure 5). We can substitute our common ratio of $\frac{1+g}{1+r}$ back in for "x". We are now left with this complex fraction, which we can condense by rewriting 1 (from the "1-x" in the denominator) as $\frac{1+r}{1+r}$ so that the denominator is now expressed in one fraction. This complex fraction can be rewritten as a regular fraction by taking the reciprocal, leaving us with (1+r) in the numerator and (r-g) in the denominator ($\frac{1+r}{r-g}$).

Video at 14:43

We have finally reached the cum dividend formula:

$$\text{price} = \lim_{n\to\infty} S_n = \frac{D_0\,(1+r_E)}{r_E - g}$$

The price of a stock, which is the present value of the infinitely many future dividends, is equal to D_0, today's dividend, multiplied by (1+ r_E), where r_E is the cost of equity, all divided by (r_E - g), where "g" is the constant growth rate of the dividends.

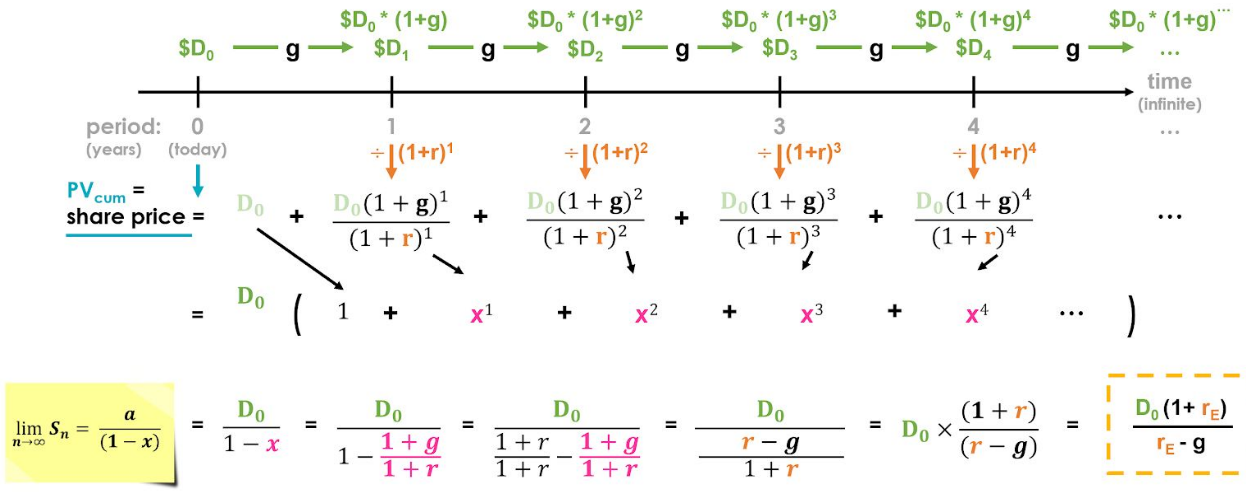


Figure 5

Video at 15:00

And just like with the ex dividend formula, the sum of the series of the infinitely many dividends can only be a finite number, if the dividends grow at a slower rate than they are being discounted. In other words, r_E must be greater than "g"; otherwise, the stock price would equal infinity. After performing a lot of math to derive both the ex dividend and cum dividend formulas, it is easy to become overwhelmed by all the equations and variables. However, keep in mind that the purpose of understanding the derivation of the formula is to help you to better understand where the Gordon Growth Model comes from, how it can be used, and and why it works. There is no magic to the Gordon Growth Model.

Video at 15:44

After all, the whole point of the formulas is to calculate the present value of all the future expected cash flows, and when you boil it down to its core, finance is really about trying to put prices on different streams of cash flows. With the ex and cum dividend formulas, we can put a price on infinite streams of dividends.

Video at 16:02

To help with understanding the formulas, there are two points to keep in mind: first, think of the ex and cum dividend formulas as two sides of the same coin. The formulas are performing the same job - calculating the price of a stock with constantly growing dividends, which is a common practice for valuing the stock of mature firms with stable growth.

Video at 16:21

The key difference is that the cum dividend formula calculates the present value right before D_0 (this period's dividend) is about to be paid, whereas the ex dividend formula calculates the present value right after D_0 has been paid. Thus, you can imagine that, for the same stock, the cum dividend present value is greater than the ex dividend present value, with the difference being the additional dividend today.

Video at 16:44

We can test this with the Nike example from earlier. With the cum dividend formula, we can see that, if Michael purchases the Nike stock in time to receive this period's \$2 dividend, the value of the stock is equal to

$$price = PV_{cum} = \frac{D_0(1+r_E)}{r_E - g} = \frac{\$2\,(1.11)}{0.11 - 0.08} = \$74$$

On the other hand, with the ex dividend formula, we can see that, if Michael just misses this year's \$2 dividend, and so the next dividend he receives is D_1, which is 8% greater than this year's dividend, then the value of the stock is

$$price = PV_{ex} = \frac{D_1}{r_E - g} = \frac{D_0(1+g)}{r_E - g} = \frac{\$2\,(1.08)}{0.11 - 0.08} = \$72$$

Video at 17:30

The difference between the cum dividend and the ex dividend stock prices is \$2, which matches this year's dividend (D_0). Is this a coincidence? Not at all! Since both the cum dividend and ex dividend formulas are calculating the present value of all the future dividends, and the difference between the stream of dividends is whether or not Michael receives the \$2 dividend today (D_0), then it makes sense that the difference in the value of the stock is the \$2 that Michael may or may not receive.

Video at 18:01

This example also leads to the second tip to keep in mind. Many students get confused between using D_0 to calculate the ex dividend stock price and using D_0 to calculate the cum dividend stock price. The key thing to remember is that, while we may use D_0 to help us find D_1 (the numerator of the ex dividend stock price), D_0 itself is not being accounted for in the ex dividend stock price. On the other hand, with the cum dividend stock price, we need D_0 , as it is actually being accounted for in the cum dividend stock price. Keeping this distinction clear in your mind is crucial to avoiding confusion between the two formulas, which, although are conceptually similar, are representing two different applications.

Video at 18:47

All right, so let's summarize everything that we have learned in this video. First, we went on a long journey to derive the ex dividend formula by drawing parallels to the sum of an infinite geometric series. We arrived at this formula: that allows us to calculate the price of stocks where the first cash flow is D_1 (the dividend to be paid next period) We followed the same process to derive the cum dividend formula, right here: where the first cash flow is D_0 (the dividend that is just about to be paid today).

Video at 19:24

Do not get the cum dividend formula confused with the modification of the ex dividend formula, where you can use D_0 to calculate D_1 in the numerator.

Video at 19:35

We also learned that we can use either formulas to calculate the price of a stock, so long as the discount rate r_E is greater than the growth rate "g".

Video at 19:47

Finally, we have emphasized that the cum dividend and ex dividend formulas are essentially two sides of the same concept: while they both calculate the price of a stock with constantly growing dividends, the cum dividend stock price is greater than the ex dividend stock price by the value of D_0 . Keep on practicing the ex dividend and cum dividend formulas, and soon, you will be valuing stocks like a pro! Thank you so much for watching!