

Video 5 - Perpetuities and Annuities: Deriving the Formulas

The following is a supplementary transcript for tutorial videos from

<https://blogs.ubc.ca/financefundamentals/>

Today, we are going to be talking about annuities and perpetuities. First, we will talk about what a perpetuity is, and how you can solve for its present value; next, we will talk about annuities, and how you can solve for their present and future values; and lastly, we will take a look at how to do all of it in Excel.

Video at 00:20

Let's start with perpetuities. A perpetuity is a stream of fixed payments delivered at fixed intervals for, well, forever. Suppose my parents decided to give me an allowance of \$10 every month and never stopped. So I would receive \$10 dollars each month, forever. What some might call a failure to launch, I would call a perpetuity.

Video at 00:40

We don't see many real-life examples of perpetuities, but some notable ones are Consols, which are British bonds that have no end date ("maturity date"), meaning that, for as long as you own it, you will receive a fixed payment. We will talk more about bonds when we discuss annuities later. A second example is preferred stock, stock that pays a fixed dividend (cash payment) to shareholders forever, or, for as long as the company is still in business.

Video at 01:05

Nothing truly goes on forever, but as long as the time horizon is sufficiently long and is undetermined, we can treat the investment as a perpetuity. Without a future date in mind, though, we can't calculate the future value of the investment, and so, we only consider the present value of a perpetuity. Let's try an example.

Video at 01:23

Suppose your pension plan will pay you \$1,000 each month for the rest of your life, with the first payment arriving at the start of next period. The time value of money principle teaches us that, if we receive the same fixed payment each period, that money is worth less and less each period in today's dollars. So, to calculate the present value of the entire pension, we basically have to

calculate the present value of every monthly payment individually and then add all of them together.

Video at 01:51

Let's say our annual discount rate is 12%. Since discount rates are quoted annually, or in APR (annual percentage rates), then we know that our effective rate, which is the monthly rate that matches with our monthly pension payments, must be

$$\text{effective monthly rate "r"} = \frac{12\%}{12 \text{ months per year}} = 1\% \text{ per month.}$$

Video at 02:09

Our first payment will have a present value of

$$\$1,000 \times \frac{1}{1+0.01} = \frac{\$1,000}{1+0.01} = \$1,000 \times (1 + 0.01)^{-1} \text{ (however you like to write it).}$$

Our second payment will have a present value of

$$\begin{aligned} & \$1,000 \times \frac{1}{1+0.01} \times \frac{1}{1+0.01} \text{ also known as,} \\ & \frac{\$1,000}{(1+0.01)^2} = \$1,000 \times (1 + 0.01)^{-2} \text{ since we must discount this cash flow for 2 months.} \end{aligned}$$

Our third payment will have a present value of

$$\frac{\$1,000}{(1+0.01)^3} = \$1,000 \times (1 + 0.01)^{-3} \text{ as we discount it back for 3 months.}$$

Video at 02:47

But didn't we say that these monthly payments go on forever? So, calculating the present value of each one would take, well, forever! Luckily, there is a simple formula we can use, instead. You might remember the formula for an infinite geometric series from high school math. Well, a perpetuity is actually an infinite geometric sequence, with each term becoming increasingly smaller as it is divided by $(1+0.01)$, which we can refer to as our "discount factor" $(\frac{1}{1+0.01})$.

Video at 03:15

The formula for this is

$$PV \text{ perpetuity} = \text{sum of infinite geometric series} = \frac{A}{1 - \text{discount factor}} = \frac{A}{1 - (\frac{1}{1+r})}$$

where "A" is our fixed payment. But, $1 - \frac{1}{1+r}$ in the denominator is just "r", so our whole formula can be written as

$$PV \text{ perpetuity} = \frac{A}{r}$$

Video at 03:31

The intuition behind this formula is that, since the present value of each payment will get smaller and smaller as time goes on, it will approach zero and, eventually, we will reach a point where the next payment adds nothing extra to our present value. Even though the theory is complicated, this formula is even easier to remember than our present value formula.

Video at 03:50

Let's try it here. We'll plug in \$1,000 for our initial payment (A), and then plug in the period rate, which is 0.01 (r), to get a present value of

$$PV \text{ perpetuity} = \frac{\$1,000}{0.01} = \$100,000 \text{ for this perpetuity.}$$

Video at 04:02

Now that we understand perpetuities, let's talk about annuities. An annuity is similar to a perpetuity in that it involves a steady inflow of fixed payments at fixed intervals, just like the allowance example we discussed; however, unlike a perpetuity, an annuity does not go on forever. We receive a finite series of payments. My parents eventually cut me off and tell me to get a job. Other examples of annuities include mortgage or pension payments, regular deposits to an investment account, and most notably, bond investments.

Video at 04:35

There are two ways of calculating the present value of an annuity. First, let's say you put \$50 into an investment account at the end of each year for 3 years to save up for your next vacation. Find the present value of those payments. We can simply solve for the present value of each payment and add them together.

Video at 04:53

But what if you put in \$50 per year for 10 years? Personally, I can't be bothered to calculate 10 present values. A faster method is to use a version of the perpetuity formula we discussed earlier. Let's compare this 10-year annuity to a situation where you would pay \$50 each year in perpetuity (Figure 1). We are interested in calculating the present value of this section, here (blue box). Basically, we want to know the value of the investment if cash flows continue forever (yellow perpetuity) minus the value of the investment if cash flows continue forever starting ten

years from now (pink perpetuity). Essentially, the value of our annuity (blue box) is the value of one perpetuity (yellow) minus another perpetuity beginning later (pink).

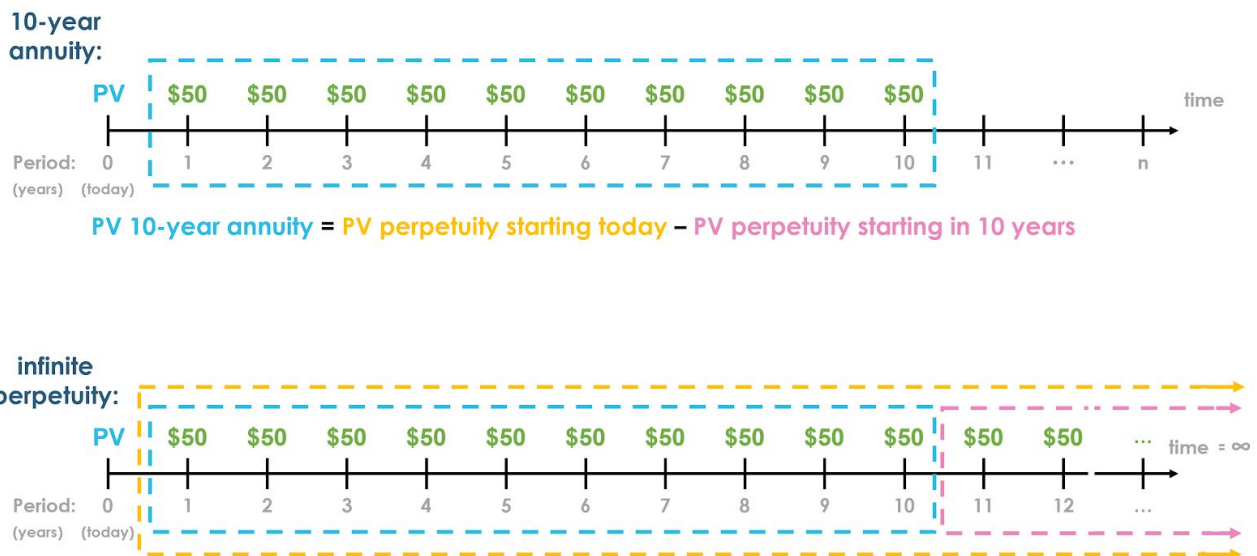


Figure 1

Video at 05:31

So, we can use this formula to solve for the present value of our annuity. Our cost of capital or discount rate is 8%. We will take

$$PV \text{ yellow perpetuity} = \frac{\$50}{0.08}$$

minus the second perpetuity (pink), which is worth $\frac{\$50}{0.08}$ ten years from now or, in today's dollars

$$PV \text{ pink perpetuity} = \frac{\$50}{0.08} \times \frac{1}{(1+0.08)^{10}} = \frac{\$50}{0.08} \times (1 + 0.08)^{-10}$$

You can think of this as either discounting the \$50 to today's terms before solving for the perpetuity, or discounting the value of the entire perpetuity to today's terms.

Video at 06:08

Solving for the equation:

$$PV \text{ yellow perpetuity} = \frac{\$50}{0.08} = \$625, \text{ and}$$

$$PV \text{ pink perpetuity} = \frac{\$50}{0.08} \times \frac{1}{(1+0.08)^{10}} = \$289.50$$

So present value of the annuity (blue) is worth

$$PV \text{ annuity} = \$625 - \$289.50 = \$335.50$$

Video at 06:25

Generally:

PV annuity = PV perpetuity starting today - PV perpetuity starting in "n" years

$$PV \text{ annuity} = \frac{A}{r} - \frac{A}{r} \times \frac{1}{(1+r)^n} = \frac{A}{r} \times \left[1 - \frac{1}{(1+r)^n} \right]$$

We can remove like terms and express this formula as

$$PV \text{ annuity} = \frac{A}{r} \times \left[1 - \frac{1}{(1+r)^n} \right] = \frac{A}{r} \times [1 - (1+r)^{-n}] = A \times \left[\frac{1-(1+r)^{-n}}{r} \right]$$

(however you like to write it)

Video at 06:35

Unlike perpetuities, we can solve for the future value of an annuity. You can choose to apply the future value formula to the present value we just calculated (PV annuity = \$335.50), or adapt the annuity formula directly. To solve for the future value, we simply multiply our present value by $(1+r)^n$. Applying this to the formula, we are left with

$$\begin{aligned} FV \text{ annuity} &= PV \text{ annuity} \times (1+r)^n \\ &= \left[\frac{A}{r} - \frac{A}{r} \times \frac{1}{(1+r)^n} \right] \times (1+r)^n \\ &= \frac{A}{r} \times (1+r)^n - \frac{A}{r} \times \frac{1}{(1+r)^n} \times (1+r)^n \\ &= \frac{A}{r} \times (1+r)^n - \frac{A}{r} \end{aligned}$$

as $(1+r)^n$ cancels in the second term (pink perpetuity).

$$\begin{aligned} &= \frac{A}{r} \times [(1+r)^n - 1] \\ &= A \times \left[\frac{(1+r)^n - 1}{r} \right] \end{aligned}$$

(however you like to write it)

Video at 07:03

This time, we are compounding the first term (yellow perpetuity) to express it in terms of its dollar value 10 years from now ($\frac{A}{r} \times (1+r)^n$), and leaving the second term (pink perpetuity in year 10 = $\frac{A}{r}$), which is already expressed in future dollars (year 10). So, in 10 years, you will have

$$FV \text{ annuity} = \frac{A}{r} \times (1+r)^n - \frac{A}{r} = \$625 \times (1+0.08)^{10} - \$625 = \$724$$

to spend on your vacation. I see some hostels in your future.

Video at 07:20

You will see the formula rearranged in several different ways, but don't be alarmed; the logic is always the same. Just use the version of the formula that you are most comfortable with.

Video at 07:30

Lastly, let's look at how we can solve for perpetuities and annuities in Excel. To calculate the present value of an annuity, we will type "`=PV`" (Figure 2) and fill in the rate ($r = 8\%$), the number of periods ($n = 10$ years), and the payment ($A = \$50$). When they ask for a future value here "[fv]", they are referring to an extra lump sum payment at the end of the investment. We don't have any payments besides our regular fixed payment, so we'll put 0. We get the same answer that we did solving by hand (Figure 5: $PV = \$335.50$).

	A	B	C	D	E	F	G	H	I
1									
2		PV of 10-year annuity							
3		Structure:	discount rate (r), number of periods (n), payment (\$C), lumpsum at the end (\$0)						
4		Calculation:	<code>=PV(0.08,10,50</code>						
5		Excel note:	the P _v is negative because we are putting money <u>away</u> into our savings account each year						

Figure 2

Video at 07:59

To calculate the future value of an annuity, we'll type "`=FV`" (Figure 3) and fill in the rate (8%), number of periods (10), the payment (\$50), and once again, ignore the option to put in a value for present value "[pv]", which refers to an additional payment made at the beginning of the investment, which we don't have here.

	A	B	C	D	E	F	G	H	I
6									
7		FV of 10-year annuity							
8		Structure:	discount rate (r), number of periods (n), payment (\$C), lumpsum at the beginning (\$0)						
9		Calculation:	<code>=FV(0.08,10,50</code>						
10		Excel note:	the F _v is negative because we are putting money <u>away</u> into our savings account each year						

Figure 3

Video at 08:14

What about solving for a perpetuity? We can manipulate the PV formula to work for perpetuities by making up an extremely large value for "n". Since the longer the time horizon, the more the present value of each cash flow approaches zero, and the more the present value of the investment approaches its value at infinity. Going back to our perpetuity example, let's plug in "=PV" (Figure 4), 0.01 for the monthly rate (r), 10,000 for the number of periods (n) -- not too large or Excel will return an error -- and \$1,000 for the payment (A). This returns the same value we calculated using our perpetuity formula (Figure 5: PV = \$100,000).

	A	B	C	D	E	F	G	H	I		
11											
12		PV of infinite perpetuity									
13		Structure:	discount rate (r), number of periods (n), payment (\$C), lumpsum at the end (\$0)								
14		Calculation:	=PV(0.01,10000,-1000,								
15		Excel note:	the payment is negative (-\$C) because we are <i>receiving</i> (not spending) money each month								

Figure 4

	A	B	C	D	E	F	G	H	I		
1											
2		PV of 10-year annuity									
3		Structure:	discount rate (r), number of periods (n), payment (\$C), lumpsum at the end (\$0)								
4		Calculation:	-\$335.50								
5		Excel note:	the PV is negative because we are putting money <i>away</i> into our savings account each year								
6											
7		FV of 10-year annuity									
8		Structure:	discount rate (r), number of periods (n), payment (\$C), lumpsum at the beginning (\$0)								
9		Calculation:	-\$724.33								
10		Excel note:	the FV is negative because we are putting money <i>away</i> into our savings account each year								
11											
12		PV of infinite perpetuity									
13		Structure:	discount rate (r), number of periods (n), payment (\$C), lumpsum at the end (\$0)								
14		Calculation:	\$100,000.00								
15		Excel note:	the payment is negative (-\$C) because we are <i>receiving</i> (not spending) money each month								

Figure 5

Video at 08:49

Remember when I said that solving for the present value of every single annuity payment would be too much effort? Well, Excel makes it simple. If you don't want to use the PV + FV formulas, you can simply set up a grid, like this one in Excel (Figure 5). We will type in our fixed payment (A = \$50 in cell C3) and our discount rate (r = 0.08 in cell C4), so that we can reference them in our formulas. Then, we will put a year in each column (cells D6:M6). I like to drag this over so Excel copies the pattern.

	A	B	C	D	E	F	G	H	I	J	K	L	M
1													
2		PV of 10-year annuity											
3		Payment (\$C)	\$50.00										
4		Discount rate (r)	0.08										
5													
6		Period (years):	0 (today)	1	2	3	4	5	6	7	8	9	10
7		Present value:											

Figure 6

Video at 09:11

Next, let's type in our present value formula (in cell D7) to solve for the present value of each payment. We will type in

$$\text{simple } PV = \frac{\text{our payment}}{(1 + \text{our discount rate})^{\text{year}}} = \frac{\$C}{(1+r)^n} = \$C \times (1+r)^{-n} = \text{cell } C3 \times (1 + \text{cell } C4)^{-\text{cell } D6}$$

$$=\$C\$3*(1+\$C\$4)^{-(D6)}$$

Remember to absolute reference the fixed payment (\$C\$3) and discount rate (\$C\$4) cells because, now, we are going to drag this over (to cell M7). See how the year value moves (D6 to M6), but our payment (\$C\$3) and discount rate (\$C\$4) don't. Now, we simply add up these ten cells (C7:M7). Notice that we get the same present value as we did using the did using the previous two methods (PV = \$335.50).

	A	B	C	D	E	F	G	H	I	J	K	L	M
1													
2		PV of 10-year annuity											
3		Payment (\$C)	\$50.00										
4		Discount rate (r)	0.08										
5													
6		Period (years):	0 (today)	1	2	3	4	5	6	7	8	9	10
7		Present value:	\$335.50	\$46.30	\$42.87	\$39.69	\$36.75	\$34.03	\$31.51	\$29.17	\$27.01	\$25.01	\$23.16

Figure 7

Video at 09:42

So, in this video, we went over perpetuities, and how to solve for their present value; next, we discussed annuities and derived the annuity formula to solve for their present and future values; and lastly, we looked at how to apply the PV and FV formulas to solving these problems.

Thanks for watching!