First Name: ______________________ Last Name: ______________________

Student-No: ______________________ Grade: ______________________

Very short answer questions

1. 4 marks Each part is worth 1 mark. Please write your answers in the boxes.

(a) Solve $(3^x)^2 = \frac{1}{5}$ for x.

Answer: $x = -\frac{1}{2}\log_3(5)$

Solution:

$$3^{2x} = \frac{1}{5}$$
$$2x = \log_3\left(\frac{1}{5}\right) = -\log_3(5)$$
$$x = -\frac{1}{2}\log_3(5)$$

(b) Compute $\lim_{t\to 1}\left(\frac{t^3+1}{e^t+1}\right)$.

Answer: $\frac{2}{e+1}$

(c) Given $H(x) = e^x(2x^3+3)$, find $\frac{dH}{dx}$.

Answer: $e^x(2x^3+3) + e^x(6x^2)$

Solution: Use product rule.

$$\frac{dH}{dx} = \frac{d}{dx}(e^x(2x^3+3)) = \frac{d}{dx}(e^x)(2x^3+3) + e^x\frac{d}{dx}(2x^3+3)$$
$$= e^x(2x^3+3) + e^x(6x^2)$$

(d) From the table below calculate $(f+2g)'(1)$:

x	6	1	-2	12	2	-1
$f(x)$	5	-2	-6	6	6	-6
$g(x)$	8	1	-1	3	-2	-3
$f'(x)$	2	2	2	6	1	10
$g'(x)$	6	3	3	3	-3	5

Answer: 8

Solution:

$$(f+2g)'(1) = f'(1) + 2g'(1) = 2 + 2 \times 3 = 8$$

Short answer questions — you must show your work

2. 8 marks Each part is worth 2 marks.

   (a) Given that $\log_x 6 = 4$, find the value of $\log_6 x^2$.

   Answer: $\frac{1}{2}$

Solution:

$$\log_x 6 = 4 \Rightarrow x^4 = 6 \Rightarrow x = 6^{1/4}$$
$$\log_6 x^2 = \log_6 \left(6^{1/4}\right)^2 = \log_6 6^{1/2} = \frac{1}{2}$$

   (b) Compute the limit $\lim_{x\to 2^-} (x+5)\frac{x-2}{|x-2|}$

   Answer: -7

Solution:

Note that since $\to 2^-$, we have $x < 2$. Therefore, $x - 2 < 0$ and $|x-2| = 2 - x$.

$$\begin{aligned}\lim_{x\to 2^-} (x+5)\frac{x-2}{|x-2|} &= \lim_{x\to 2^-} (x+5)\frac{x-2}{2-x} = \lim_{x\to 2^-} ((x+5)\times(-1)) \\ &= \lim_{x\to 2^-} (-x-5) = -7\end{aligned}$$

   (c) Find a such that $f(x) = \begin{cases} \frac{1}{ax}, & x \leq 1 \\ \sqrt{x^2+3}, & x > 1 \end{cases}$ is continuous at $x = 1$.

   Answer: $a = \frac{1}{2}$

Solution:

For continuity at $x = 1$ we need $\lim_{x\to 1^+} f(x) = \lim_{x\to 1^-} f(x) = f(x)$.

$$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} \sqrt{x^2+3} = 2$$
$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^+} \frac{1}{ax} = \frac{1}{a}$$
$$f(1) = \frac{1}{a}$$

$$\Rightarrow \frac{1}{a} = 2 \Rightarrow a = \frac{1}{2}$$

(d) Find the equation of the tangent line to $f(x) = 2\sin(x) + x$ at $x = \pi/2$.

Answer: $y = x + 2$

Solution:

$$\frac{df}{dx} = 2\cos(x) + 1 \Rightarrow f'(\pi/2) = 2\cos(\pi/2) + 1 = 1$$
$$x = \pi/2 \Rightarrow f(\pi/2) = 2\sin(\pi/2) + \pi/2 = 2 + \pi/2$$

So the equation of the tangent line to $f(x) = 2\sin(x) + x$ at $x = \pi/2$ is

$$y - (2 + \pi/2) = x - \pi/2 \Rightarrow y = x - \pi/2 + 2 + \pi/2 = x + 2$$

Long answer question — you must show your work

3. 4 marks A manufacturer sells 50 tables a month at the price of \$300 each. For each \$8 decrease in price, he can sell 2 more tables. Their factory costs \$5,000 per month to operate and each table costs an additional \$50 to make.

Note: in this problem you are ONLY setting up the equations. You do NOT have to solve for break even values or any optimal production values.

(a) Find the linear demand equation for the tables. Use the notation p for the unit price and q for the monthly demand.

Answer: $p = -4q + 500$

Solution: A data point is $(q, p) =$ (quantity, price). So two points are $(50, 300)$ and $(52, 292)$.

$$p = mq + b \quad (1)$$
$$300 = 50m + b \quad (2)$$
$$292 = 52m + b \quad (3)$$

We can solve equation (2) for b to find $b = 300 - 50m$. We can then plug this in equation (3) to find m:

$$292 - 52m + 300 - 50m \quad (4)$$
$$292 - 300 = -50m + 52m \quad (5)$$
$$-8 = 2m \quad (6)$$
$$-4 = m \Rightarrow b = 300 - 50m = 300 - 50 \times (-4) = 300 + 200 = 500 \quad (7)$$

Now that we've found m and b, we can plug them back in equation (1) to find the demand equation

$$p = -4q + 500$$

(b) Find the cost function, $C = C(q)$, for producing q tables per month.

Answer: $C(q) = 5000 + 50q$

Solution: No work is needed

(c) Find the monthly revenue function, $R = R(q)$.

Answer: $R(q) = -4q^2 + 500q$

Solution: $R = p \cdot q = q \cdot (-4q + 500) = -4q^2 + 500q$

4. [4 marks] Use the *definition of the derivative*, i.e.the limit process, to find the slope of the tangent line to the graph of $y = \sqrt[3]{x}$ at $x = 8$. Please put a box around your final answer.

Solution:

$$\begin{aligned}
f'(2) &= \lim_{x\to 2} \frac{f(x) - f(8)}{x - 8} = \lim_{x\to 2} \frac{\sqrt[3]{x} - \sqrt[3]{8}}{x - 8} \\
&= \lim_{x\to 2} \frac{\sqrt[3]{x} - 2}{x - 8} \\
&= \lim_{x\to 2} \left(\frac{\sqrt[3]{x} - 2}{x - 8} \frac{x^{2/3} + 2^2 + 2x^{1/3}}{x^{2/3} + 2^2 + 2x^{1/3}} \right) \\
&= \lim_{x\to 2} \left(\frac{(x^{1/3} - 2)(x^{2/3} + 2^2 + (2x)^{1/3})}{(x - 8)(x^{2/3} + 2^2 + 2x^{1/3})} \right) \\
&= \lim_{x\to 2} \left(\frac{x - 8}{(x - 8)(x^{2/3} + 2^2 + 2x^{1/3})} \right) \\
&= \lim_{x\to 2} \left(\frac{1}{x^{2/3} + 2^2 + 2x^{1/3}} \right) \\
&= \frac{1}{8^{2/3} + 2^2 + 2 \times 8^{1/3}} \\
&= \frac{1}{(2^3)^{2/3} + 2^2 + 2 \times (2^3)^{1/3}} \\
&= \frac{1}{2^2 + 2^2 + 2 \times 2} \\
&= \frac{1}{12}
\end{aligned}$$