Week 2 - examples

May 2016

Example 1. Continuity

Find the points where $f(x)$ is not continuous.

$$f(x) = \begin{cases} \dfrac{x^2+1}{x^2-2x-3} & x \leq 1 \\ \dfrac{|x-3|}{2x-6}(2-x) & 1 < x < 3 \\ \dfrac{1}{2} & x = 3 \\ \dfrac{\exp(x-3)}{x^2-20} & 3 < x \end{cases}$$

Note: $\exp(x-3)$ is another notation for e^{x-3}; i.e. $\exp(x-3) = e^{x-3}$.

Solution: When given a function with different definitions on different intervals, we should

- verify continuity on each interval,
- check continuity at the endpoints of the intervals.

So here, we first check continuity on $(-\infty, 1)$:

$$x \leq 1 \Rightarrow f(x) = \frac{x^2+1}{x^2-2x-3}$$

Here f is a rational function; i.e. both the numerator and denominator of f are polynomials. We know that rational functions are continuous for all x in their domain. That is for all x where the denominator is not zero. So we need to find the roots of the denominator:

$$x^2 - 2x - 3 = 0 \Rightarrow x = 3, \; x = -1$$

Now what we need to be careful about is that we started with the assumption that $x \leq 1$; $x = 3$ is not in the domain we are considering right now. **So we've found that on the interval $x < 1$, f is continuous everywhere except at $x = -1$.**

Now we have to check continuity at $x = 1$. To do this, we need to show:

$$\lim_{x \to 1} f(x) = f(1)$$

This is equivalent to

$$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^-} f(x) = f(1)$$

Now the easier step is perhaps to find $f(1)$:

$$f(1) = \left(\frac{x^2+1}{x^2-2x-3}\right)_{x=1} = \frac{1+1}{1-2-3} = \frac{2}{-4} = -\frac{1}{2}$$

Next we find $\lim_{x\to 1^+} f(x)$; i.e. the limit of $f(x)$ as x approaches 1 and $x > 1$. Looking at the definition of $f(x)$ we find that when x approaches 1 and $x > 1$

$$f(x) = \frac{|x-3|}{2x-6}(2-x)$$

Therefore,

$$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} \frac{|x-3|}{2x-6}(2-x)$$

Now I have to be careful about the absolute function: $|x-3|$.
Reminder:

$$|a| = \begin{cases} a & a \geq 0 \\ -a & a < 0 \end{cases}$$

Now since $x \to 1^+$, we know that $(x-3) \to -2^+$. Therefore, when $x \to 1^+$, we have $|x-3| = -(x-3) = -x+3$.
We can now evaluate the right-hand limit:

$$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} \frac{|x-3|}{2x-6}(2-x) = \lim_{x\to 1^+} \frac{(-x+3)(2-x)}{2x-6}$$

We're taking the limit of a rational function. After making sure that the denominator is not zero at $x = 1$, we can evaluate the limit by substitution:

$$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} \frac{(-x+3)(2-x)}{2x-6} = \frac{2\times(1)}{2-6} = -\frac{2}{4} = -\frac{1}{2}$$

So far we have

$$\lim_{x\to 1^+} f(x) = f(1) = -\frac{1}{2}$$

The last step of verifying continuity at $x = 1$ is to find $\lim_{x\to 1^-} f(x)$; i.e. the limit of $f(x)$ as x approaches 1 and $x < 1$. Now looking back at definition of $f(x)$, we find that when $x \to 1^-$

$$f(x) = \frac{x^2+1}{x^2-2x-3}$$

Again, this is a rational function and it is continuous everywhere in its domain. Since the denominator is not zero at $x = 1$, the limit can be evaluated by substitution:

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} \frac{x^2+1}{x^2-2x-3} = \frac{1+1}{1-2-3} = \frac{2}{-4} = -\frac{1}{2}$$

So we have shown that

$$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^-} f(x) = f(1) = -\frac{1}{2}$$

Therefore $f(x)$ is continuous at $x = 1$

Next, we should examine continuity on $1 < x < 3$. When $1 < x < 3$ we have

$$f(x) = \frac{|x-3|}{2x-6}(2-x)$$

Again we have to be careful about the absolute function: $|x-3|$. Since $x < 3$, we know that $x - 3 < 0$. Therefore, $|x-3| = -(x-3) = -x+3$. So we have

$$f(x) = \frac{(-x+3)(2-x)}{2x-6}$$

Now we have a rational function. Rational functions are continuous everywhere in their domain. Thus we find the points where the denominator becomes zero:

$$2x - 6 = 0 \Rightarrow x = 3$$

Since we're interested in $1 < x < 3$, we can therefore conclude that $f(x)$ is continuous on this interval.

Next we examine continuity at $x = 3$. For $f(x)$ to be continuous at $x = 3$ we need

$$\lim_{x\to 3^+} f(x) = \lim_{x\to 3^-} f(x) = f(3)$$

Based on the definition of the function we have $f(3) = 1/2$. So we need to find the left-hand and right-hand limits at $x = 3$.

Notice that considering the left-hand limit, x is approaching 3 and $x < 3$. Therefore,

$$\lim_{x\to 3^-} f(x) = \lim_{x\to 3^-} \frac{|x-3|}{2x-6}(2-x)$$

Once again, we need to be careful about the absolute function: $|x-3|$. We note that since $x \to 3^-$ we have

$$x < 3 \to x - 3 < 0 \Rightarrow |x-3| = -(x-3) = -x+3.$$

Going back to the left-hand limit we have

$$\lim_{x\to 3^-} f(x) = \lim_{x\to 3^-} \frac{(-x+3)(2-x)}{2x-6}$$

Again, we're taking the limit of a rational function and therefore this function is continuous everywhere in its domain. However, we know that the denominator is 0 at $x = 3$ and so $x = 3$ is not in the domain of this rational function. Note that the numerator is also 0 at $x = 3$. So we need to factor and simplify to find the limit:

$$\lim_{x\to 3^-} f(x) = \lim_{x\to 3^-} \frac{(-x+3)(2-x)}{2x-6} = \lim_{x\to 3^-} \frac{-1\times(x-3)(2-x)}{2\times(x-3)} = \lim_{x\to 3^-} \frac{-1\times(2-x)}{2} = \frac{1}{2}$$

Finally, we should evaluate the right-hand limit; that is as x approaches 3 and $x > 3$. So we have

$$\lim_{x\to 3^+} f(x) = \lim_{x\to 3^+} \frac{\exp(x-3)}{x^2-20}$$

We know that both numerator and denominator have a right-hand limit when $x \to 3^+$:

$$\lim_{x\to 3^+} \exp(x-3) = \exp(3-3) = 1$$
$$\lim_{x\to 3^+} (x^2-20) = 9-20 = -11$$

Therefore,

$$\lim_{x\to 3^+} f(x) = \lim_{x\to 3^+} \frac{\exp(x-3)}{x^2-20} = \frac{\lim_{x\to 3^+}\exp(x-3)}{\lim_{x\to 3^+}(x^2-20)} = -\frac{1}{11}$$

So we have found that

$$\lim_{x\to 3^-} f(x) = f(3) = \frac{1}{2}$$
$$\lim_{x\to 3^+} f(x) = -\frac{1}{11}$$

Therefore $f(x)$ is continuous from the left at $x = 3$. But it is not continuous from the right. So $f(x)$ is not continuous at $x = 3$.

Finally, we consider the interval $3 < x$. Here we have

$$f(x) = \frac{\exp(x-3)}{x^2-20}$$

Now we know that exponential functions are continuous everywhere in their domain. Similarly, the denominator is a polynomial and is continuous everywhere. So $f(x)$ is continuous where the denominator is not zero. Therefore, We need to find the roots of the denominator:

$$x^2 - 20 = 0 \Rightarrow x = -\sqrt{20}, x = \sqrt{20}$$

Notice that we have assumed $3 < x$. Also, although I may not have a calculator to find $\sqrt{20}$, I can see that since $9 < 20$, indeed $3 < \sqrt{20}$. **Therefore, $f(x)$ is not continuous at $x = \sqrt{20}$.**

the graph of function $f(x)$ is provided in Fig. 1. Can you find the points where the function is not differentiable?

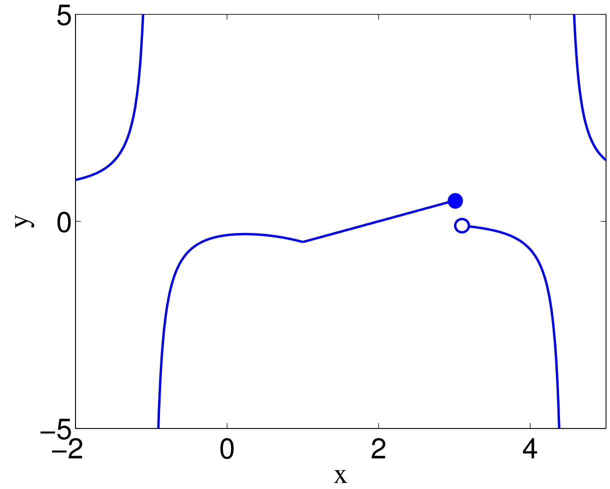

Figure 1: Graph of $f(x)$ on $-2 < x < 5$.

Example 2. Tangent lines and derivatives

Given $f(x) = x^3 - 6x^2 + \pi$,

a. Find all the points where the tangent line is horizontal and give the equation of the tangent line(s).

b. Find the point(s) where tangent line has slope -12 and give the equation of the tangent line.

Solution: To find the slope of the tangent line at any point in the domain of a function, we should evaluate the derivative of f at that point; i.e. if m is the slope of the tangent line at $x = a$, we have $m = f'(a)$. So let's find $f'(x)$:

$$f'(x) = \frac{d}{dx}(x^3 - 6x^2 + \pi) = \frac{d}{dx}(x^3) - \frac{d}{dx}(6x^2) + \frac{d}{dx}(\pi) = 3x^2 - 6 \times 2x + 0 = 3x^2 - 12x$$
$$f'(x) = 3x^2 - 12x$$

Alternatively, we could have found the derivative function using the deifinition of the first detivative:

$$\begin{aligned} f'(x) &= \lim_{h\to h} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{3(x+h)^3 - 6(x+h)^2 + \pi - (x^3 - 6x^2 + \pi)}{h} \\ &= \lim_{h\to 0} \frac{(x^3 + 3x^2h + 3xh^2 + h^3) - 6(x^2 + 2xh + h^2) + \pi - (x^3 - 6x^2 + \pi)}{h} \\ &= \lim_{h\to 0} \frac{3x^2h + 3xh^2 + h^3 - 12xh - 6h^2}{h} \\ &= \lim_{h\to 0}(3x^2 + 3xh + h^2 - 12x - 6h) \\ &= 3x^2 - 12x \end{aligned}$$

So at the point $x = a$ the slope of the tangent line is $f'(a) = 3a^2 - 12a$.

I also could have solved for the slope of the tangent line (that is the value of the derivative function or the instantaneous rate of change of f) at $x = a$:

$$\begin{aligned}
f'(a) &= \lim_{x\to a} \frac{f(x) - f(a)}{x - a} = \lim_{x\to a} \frac{x^3 - 6x^2 + \pi - (a^3 - 6a^2 + \pi)}{x - a} \\
&= \lim_{x\to a} \frac{x^3 - a^3 - 6x^2 + 6a^2 + \pi - \pi}{x - a} \\
&= \lim_{x\to a} \frac{(x^3 - a^3) - 6(x^2 - a^2)}{x - a} \\
&= \lim_{x\to a} \frac{(x - a)(x^2 + ax + a^2) - 6(x - a)(x + a)}{x - a} \\
&= \lim_{x\to a} (x^2 + ax + a^2) - 6(x + a) \\
&= a^2 + a^2 + a^2 - 6(2a) \\
&= 3a^2 - 12a
\end{aligned}$$

Now for part (a) we are looking for points where the tangent line is horizontal, i.e. the slope is 0.

$$3a^2 - 12a = 0 \Rightarrow a(3a - 12) = 0 \Rightarrow a = 0, a = \frac{12}{3} = 4$$

We have found the slope of the tangent lines. We need to find a point on the line: $(a, f(a))$.

$$\begin{aligned}
&a = 0 \Rightarrow f(0) = \pi \\
&a = 4 \Rightarrow f(4) = 4^3 - 6 \times 4^2 + \pi = 64 - 96 + \pi = -32 + \pi
\end{aligned}$$

Now the equation of horizontal lines is given by $y = f(a)$; i.e. there are two points where the tangent line is horizontal:

$$\begin{aligned}
&x = 0, \text{ euqation of tangent line: } y = \pi \\
&x = 4, \text{ euqation of tangent line: } y = -32 + \pi
\end{aligned}$$

For part (b) we should look for poitns where the slope of the tangent line is -12. That is $f'(a) = 5$

$$\begin{aligned}
&3a^2 - 12a = -12 \Rightarrow a^2 - 4a + 4 = 0 \Rightarrow a = \frac{2 \pm \sqrt{2^2 - 4 \times 1}}{1} = 2 \\
&f(2) = 2^3 - 6 \times 2^2 + \pi = 8 - 24 + \pi = -16 + \pi
\end{aligned}$$

So with the slope of the line (that is -12) and a point on it (that is $(2, -16 + \pi)$), we can write the equation of the tangent line:

$$\begin{aligned}
&y - f(a) = f'(a)(x - a) \\
&y - (-16 + \pi) = -12(x - 2) \\
&y = -12x + 24 - 16 + \pi = -12x + 8 + \pi
\end{aligned}$$

The tangent line at $x = 2$ has the the slope -12. The equation of the tangent line at $x = 2$ is $y = -12x + 8 + \pi$.