6.8 Exponential Models

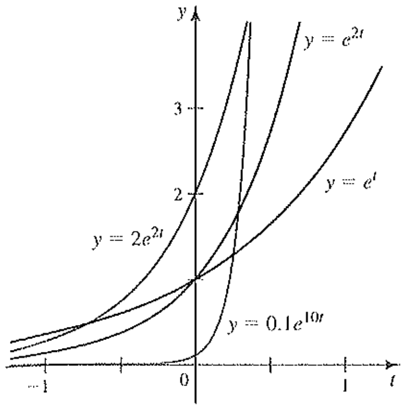

FIGURE 6.69

The uses of exponential functions are wide-ranging. In this section, you will see them applied to problems in finance, medicine, ecology, biology, economics, pharmacokinetics, anthropology, and physics.

Exponential Growth

Exponential growth models use functions of the form $y(t) = Ce^{kt}$, where C is a constant and the **rate constant** k is positive (Figure 6.69).

If we start with the exponential growth function $y(t) = Ce^{kt}$ and take its derivative, we find that

$$\frac{dy}{dt} = \frac{d}{dt}(Ce^{kt}) = C \cdot ke^{kt} = k(\underbrace{Ce^{kt}}_{y})$$

that is, $\frac{dy}{dt} = ky$. Here is the first insight about exponential functions: *Their rate of change is proportional to their value.* If y represents a population, then $\frac{dy}{dt}$ is the **growth rate** with units such as people/month or cells/hr. And if y is an exponential function, then the more people present, the faster the population grows.

➤ The derivative $\frac{dy}{dt}$ is the *absolute* growth rate but is usually more simply called the *growth rate*.

Another way to talk about growth rates is to use the **relative growth rate**, which is the growth rate divided by the current value of that quantity—that is, $\frac{1}{y}\frac{dy}{dt}$. For example, if y is a population, the relative growth rate is the fraction or percentage by which the population grows each unit of time. Examples of relative growth rates are *5% per year* or *a factor of 1.2 per month*. Therefore, when the equation $\frac{dy}{dt} = ky$ is written in the form $\frac{1}{y}\frac{dy}{dt} = k$, it has another interpretation. It says that *a quantity that grows exponentially has a constant relative growth rate*. Constant relative or percentage change is the hallmark of exponential growth.

➤ A consumer price index that increases at a constant rate of 4% per year increases exponentially. A currency that is devalued at a constant rate of 3% per month decreases exponentially. By contrast, linear growth is characterized by constant absolute growth rates, such as 500 people per year or \$400 per month.

EXAMPLE 1 **Linear vs. exponential growth** Suppose the population of the town of Pine is given by $P(t) = 1500 + 125t$, while the population of the town of Spruce is given by $S(t) = 1500e^{0.1t}$, where $t \geq 0$ is measured in years. Find the growth rates and the relative growth rates of the two towns.

SOLUTION Note that Pine grows according to a linear function, while Spruce grows exponentially (Figure 6.70). The growth rate of Pine is $\frac{dP}{dt} = 125$ people/yr, which is constant for all times. The growth rate of Spruce is

$$\frac{dS}{dt} = 0.1(\underbrace{1500e^{0.1t}}_{S(t)}) = 0.1S(t),$$

showing that the growth rate is proportional to the population. The relative growth rate of Pine is $\frac{1}{P}\frac{dP}{dt} = \frac{125}{1500 + 125t}$, which decreases in time. The relative growth rate of Spruce is

$$\frac{1}{S}\frac{dS}{dt} = \frac{0.1 \cdot 1500e^{0.1t}}{1500e^{0.1t}} = 0.1,$$

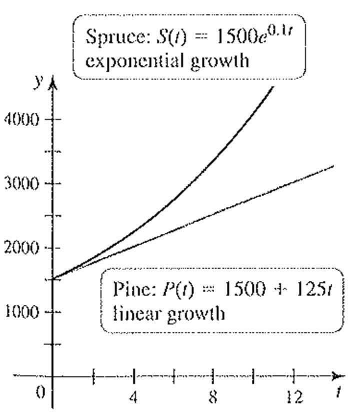

FIGURE 6.70

which is constant for all times. In summary, the linear population function has a *constant absolute growth rate*, while the exponential population function has a *constant relative growth rate*.

Related Exercises 9–10 ◀

QUICK CHECK 1 Population A increases at a constant rate of 4%/yr. Population B increases at a constant rate of 500 people/yr. Which population exhibits exponential growth? What kind of growth is exhibited by the other population? ◀

The rate constant k in $y(t) = Ce^{kt}$ determines the growth rate of the exponential function. We adopt the convention that $k > 0$; then it is clear that $y(t) = Ce^{kt}$ describes exponential growth and $y(t) = Ce^{-kt}$ describes exponential decay, to be discussed shortly. For problems that involve time, the units of k are time^{-1}; for example, if t is measured in months, the units of k are month^{-1}. In this way, the exponent kt is dimensionless (without units).

➤ The unit time^{-1} is read *per unit time*. For example, month^{-1} is read *per month*.

Unless there is good reason to do otherwise, it is customary to take $t = 0$ as the reference point for time. Notice that with $y(t) = Ce^{kt}$, we have $y(0) = C$. Therefore, C has a simple meaning: It is the **initial value** of the quantity of interest, which we denote y_0. In the examples that follow, two pieces of information are typically given: the initial condition and clues for determining the rate constant k. The initial condition and the rate constant determine an exponential growth function completely.

Exponential Growth Functions

Exponential growth is described by functions of the form $y(t) = y_0e^{kt}$. The initial value of y at $t = 0$ is $y(0) = y_0$ and the **rate constant** $k > 0$ determines the rate of growth. Exponential growth is characterized by a constant relative growth rate.

Because exponential growth is characterized by a constant relative growth rate, the time required for a quantity to double (a 100% increase) is constant. Therefore, one way to describe an exponentially growing quantity is to give its **doubling time**. To compute the time it takes for the function $y(t) = y_0e^{kt}$ to double in value, say from y_0 to $2y_0$, we find the value of t that satisfies

$$y(t) = 2y_0 \quad \text{or} \quad y_0e^{kt} = 2y_0.$$

➤ Note that the initial value y_0 appears on both sides of this equation. It may be canceled, meaning that the doubling time is independent of the initial condition: *The doubling time is constant for all t.*

Canceling y_0 from the equation $y_0e^{kt} = 2y_0$ leaves the equation $e^{kt} = 2$. Taking logarithms of both sides, we have $\ln e^{kt} = \ln 2$, or $kt = \ln 2$, which has the solution $t = \frac{\ln 2}{k}$. We denote this doubling time T_2 so that $T_2 = \frac{\ln 2}{k}$. If y increases exponentially, the time it takes to double from 100 to 200 is the same as the time it takes to double from 1000 to 2000.

QUICK CHECK 2 Verify that the time needed for $y(t) = y_0e^{kt}$ to double from y_0 to $2y_0$ is the same as the time needed to double from $2y_0$ to $4y_0$. ◀

DEFINITION Doubling Time

The quantity described by the function $y(t) = y_0e^{kt}$ for $k > 0$ has a constant doubling time of $T_2 = \frac{\ln 2}{k}$, with the same units as t.

EXAMPLE 2 World population Human population growth rates vary geographically and fluctuate over time. The overall growth rate for world population peaked at an annual rate of 2.1% per year in the 1960s. Assume a world population of 6.0 billion in 1999 ($t = 0$) and 6.9 billion in 2009 ($t = 10$).

a. Find an exponential growth function for the world population that fits the two data points.

b. Find the doubling time for the world population using the model in part (a).

c. Find the (absolute) growth rate $y'(t)$ and graph it for $0 \le t \le 50$.

d. How fast was the population growing in 2010 ($t = 11$)?

World population

1804	1 billion
1927	2 billion
1960	3 billion
1974	4 billion
1987	5 billion
1999	6 billion
2011	7 billion (proj.)

It is a common mistake to assume that if the annual growth rate is 1.4% per year, then $k = 1.4\% = 0.014\ \text{yr}^{-1}$. The rate constant k must be calculated, as it is in Example 2 to give $k = 0.013976$. For larger growth rates, the difference between k and the growth rate is greater.

SOLUTION

a. Let $y(t)$ be world population measured in billions of people t years after 1999. We use the growth function $y(t) = y_0e^{kt}$, where y_0 and k must be determined. The initial value is $y_0 = 6$ (billion). To determine the rate constant k, we use the fact that $y(10) = 6.9$. Substituting $t = 10$ into the growth function with $y_0 = 6$ implies

$$y(10) = 6e^{10k} = 6.9.$$

Solving for k yields the rate constant $k = \dfrac{\ln(6.9/6)}{10} \approx 0.013976 \approx 0.014\ \text{yr}^{-1}$.

Therefore, the growth function is

$$y(t) = 6e^{0.014t}.$$

b. The doubling time of the population is

$$T_2 = \frac{\ln 2}{k} \approx \frac{\ln 2}{0.014} \approx 50 \text{ years}.$$

c. Working with the growth function $y(t) = 6e^{0.014t}$, we find that

$$y'(t) = 6(0.014)e^{0.014t} = 0.084e^{0.014t}$$

which has units of *billions of people/yr*. As shown in Figure 6.71 the growth rate itself increases exponentially.

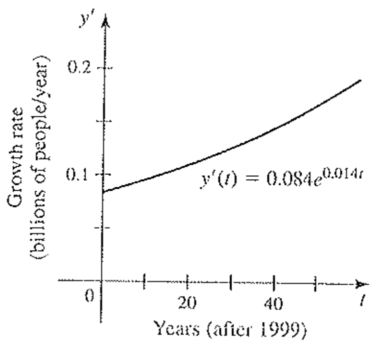

FIGURE 6.71

d. In 2010 ($t = 11$), the growth rate was

$$y'(11) = 0.084e^{(0.014)(11)} \approx 0.098 \text{ billion people/yr}$$

or roughly 98 million people/yr. *Related Exercises 11–16*

Converted to a daily rate (dividing by 365), the world population in 2010 increased at a rate of roughly 268,000 people per day.

QUICK CHECK 3 Assume $y(t) = 100e^{0.05t}$. By what percentage does y increase when t increases by 1 unit?

A Financial Model Exponential functions are used in many financial applications, several of which are explored in the exercises. For now, consider a simple savings account in which an initial deposit earns interest that is reinvested in the account. Interest payments are made on a regular basis (for example, annually, monthly, daily) or interest may be compounded continuously. With continuous compounding, the balance in the account increases exponentially at a rate that can be determined from the advertised **annual percentage yield** (or **APY**) of the account. Assuming that no additional deposits are made, the balance in the account is given by the exponential growth function $y(t) = y_0e^{kt}$, where y_0 is the initial deposit, t is measured in years, and k is determined by the annual percentage yield.

EXAMPLE 3 **Continuous compounding** The APY of a savings account is the percentage increase in the balance over the course of a year. Suppose you deposit \$500 in a savings account that has an APY of 6.18% per year with continuous compounding. Assume that the interest rate remains constant and that no additional deposits or withdrawals are made. How long will it take for the balance to reach \$2500?

The rate constant k, in this case $0.06 = 6\%$, is the factor by which the balance increases if interest is compounded once at the end of the year. It is often advertised by banks as the *annual percentage rate* (or APR). If the balance increases by 6.18% in one year, it increases by a factor of 1.0618 in one year.

SOLUTION Because the balance grows by a fixed percentage every year, it grows exponentially. Letting $y(t)$ be the balance t years after the initial deposit of $y_0 = \$500$, we have $y(t) = y_0e^{kt}$, where the rate constant k must be determined. Note that if the initial balance is y_0, one year later the balance is 6.18% more, or

$$y(1) = 1.0618\,y_0 = y_0e^k.$$

Solving for k, we find that the rate constant is

$$k = \ln 1.0618 \approx 0.060\,\text{yr}^{-1}.$$

Therefore, the balance at any time $t \geq 0$ is $y(t) = 500e^{0.060t}$. To determine the time required for the balance to reach \$2500, we solve the equation

$$y(t) = 500e^{0.060t} = 2500.$$

Dividing by 500 and taking the natural logarithm of both sides yields

$$0.060t = \ln 5.$$

The balance reaches \$2500 in $t = (\ln 5)/0.060 \approx 26.8$ yr. *Related Exercises 11–16* ◀

Resource Consumption Among the many resources that people use, energy is certainly one of the most important. The basic unit of energy is the **joule** (J), roughly the energy needed to lift a 0.1-kg object (say an orange) 1 m. The *rate* at which energy is consumed is called **power**. The basic unit of power is the **watt** (W), where 1 W = 1 J/s. If you turn on a 100-W lightbulb for 1 min, you use energy at a rate of 100 J/s and use a total of $100\,\text{J/s} \cdot 60\,\text{s} = 6000\,\text{J}$ of energy.

A more useful measure of energy for large quantities is the **kilowatt-hour** (kWh). A kilowatt is 1000 W or 1000 J/s. So if you consume energy at the rate of 1 kW for 1 hr (3600 s), you use a total of $1000\,\text{J/s} \cdot 3600\,\text{s} = 3.6 \times 10^6\,\text{J}$, which is 1 kWh. A person running for 1 hr consumes roughly 1 kWh of energy. A typical house uses on the order of 1000 kWh of energy in a month.

Assume that the total energy used (by a person, machine, or city) is given by the function $E(t)$. Because the power $P(t)$ is the rate at which energy is used, we have $P(t) = E'(t)$. Using the ideas of Section 6.1, the total amount of energy used between the times $t = a$ and $t = b$ is

$$\text{total energy used} = \int_a^b E'(t)\,dt = \int_a^b P(t)\,dt.$$

We see that energy is the area under the power curve. With this background, we can investigate a situation in which the rate of energy consumption increases exponentially.

EXAMPLE 4 Energy consumption At the beginning of 2006, the rate of energy consumption for the city of Denver was 7000 megawatts (MW), where 1 MW = 10^6 W. That rate was expected to increase at an annual growth rate of 2% per year.

a. Find the function that gives the power or rate of energy consumption for all times after the beginning of 2006.

b. Find the total amount of energy used during the year 2010.

c. Find the function that gives the total (cumulative) amount of energy used by the city between 2006 and any time $t \geq 0$.

SOLUTION

In one year, the power function increases by 2% or by a factor of 1.02.

a. Let $t \geq 0$ be the number of years after the beginning of 2006, and let $P(t)$ be the power function that gives the rate of energy consumption at time t. Because P increases at a constant rate of 2% per year, it increases exponentially. Therefore, $P(t) = P_0e^{kt}$, where $P_0 = 7000$ MW. We determine k as before by setting $t = 1$; after one year the power is

$$P(1) = P_0e^k = 1.02P_0.$$

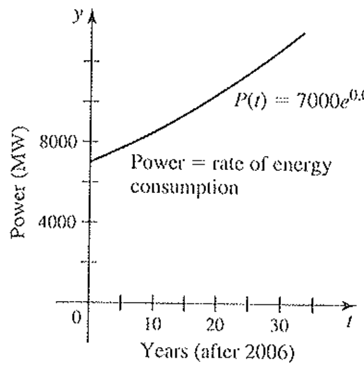

FIGURE 6.72

Canceling P_0 and solving for k, we find that $k = \ln(1.02) \approx 0.0198$. Therefore, the power function (Figure 6.72) is

$$P(t) = 7000e^{0.0198t} \quad \text{for } t \geq 0.$$

b. The entire year 2010 corresponds to the interval $4 \leq t \leq 5$. Substituting $P(t) = 7000e^{0.0198t}$, the total energy used in 2010 was

$$\begin{aligned}\int_4^5 P(t)\,dt &= \int_4^5 7000e^{0.0198t}\,dt && \text{Substitute for } P(t).\\ &= \frac{7000}{0.0198}e^{0.0198t}\Big|_4^5 && \text{Fundamental Theorem}\\ &\approx 7652 \text{ MW-years} && \text{Evaluate.}\end{aligned}$$

Because the units of P are MW and t is measured in yr, the units of energy are MW-years. To convert to MWh, we multiply by 8760 hr/yr to get the total energy of about 6.7×10^7 MWh (or 6.7×10^{10} kWh).

c. The total energy used between $t = 0$ and at any future time t is given by the future value formula (Section 6.1):

$$E(t) = E(0) + \int_0^t E'(s)\,ds = E(0) + \int_0^t P(s)\,ds.$$

Assuming $t = 0$ corresponds to the beginning of 2006, we take $E(0) = 0$. Substituting again for the power function P, the total energy at time t is

$$\begin{aligned}E(t) &= E(0) + \int_0^t P(s)\,ds\\ &= 0 + \int_0^t 7000e^{0.0198s}\,ds && \text{Substitute for } P(s) \text{ and } E(0).\\ &= \frac{7000}{0.0198}e^{0.0198s}\Big|_0^t && \text{Fundamental Theorem}\\ &\approx 353{,}535(e^{0.0198t} - 1) \text{ MW-yr} && \text{Evaluate.}\end{aligned}$$

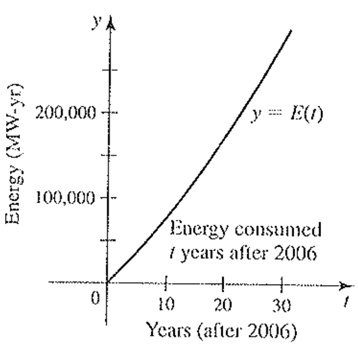

FIGURE 6.73

As shown in Figure 6.73, when the rate of energy consumption increases exponentially, the total amount of energy consumed also increases exponentially.

Related Exercises 11–16 ◀

Exponential Decay

Everything you have learned about exponential growth carries over directly to exponential decay. A function that decreases exponentially has the form $y(t) = y_0e^{-kt}$, where $y_0 = y(0)$ is the initial value and $k > 0$ is the rate constant.

Exponential decay is characterized by a constant relative decay rate and by a constant half-life. For example, radioactive plutonium has a half-life of 24,000 years. An initial sample of 1 mg decays to 0.5 mg after 24,000 years and to 0.25 mg after 48,000 years. To compute the half-life, we determine the time required for the quantity $y(t) = y_0e^{-kt}$ to reach one half of its current value; that is, we solve $y_0e^{-kt} = y_0/2$ for t. Canceling y_0 and taking logarithms of both sides, we find that

$$e^{-kt} = \frac{1}{2} \Rightarrow -kt = \ln\left(\frac{1}{2}\right) = -\ln 2 \Rightarrow t = \frac{\ln 2}{k}.$$

The half-life is given by the same formula as the doubling time.

QUICK CHECK 4 If a quantity decreases by a factor of 8 every 30 yr, what is its half-life? ◀

Exponential Decay Functions

Exponential decay is described by functions of the form $y(t) = y_0 e^{-kt}$. The initial value of y is $y(0) = y_0$, and the rate constant $k > 0$ determines the rate of decay. Exponential decay is characterized by a constant relative decay rate. The constant half-life is $T_{1/2} = \frac{\ln 2}{k}$, with the same units as t.

Radiometric Dating A powerful method for estimating the age of ancient objects (for example, fossils, bones, meteorites, and cave paintings) relies on the radioactive decay of certain elements. A common version of radiometric dating uses the carbon isotope C-14, which is present in all living matter. When a living organism dies, it ceases to replace C-14, and the C-14 that is present decays with a half-life of about $T_{1/2} = 5730$ yr. Comparing the C-14 in a living organism to the amount in a dead sample provides an estimate of its age.

EXAMPLE 5 **Radiometric dating** Researchers determine that a fossilized bone has 30% of the C-14 of a live bone. Estimate the age of the bone. Assume a half-life for C-14 of 5730 yr.

SOLUTION The exponential decay function $y(t) = y_0 e^{-kt}$ may be invoked as it applies to all decay processes with a constant half-life. By the half-life formula, $T_{1/2} = (\ln 2)/k$. Substituting $T_{1/2} = 5730$ yr, the rate constant is

$$k = \frac{\ln 2}{T_{1/2}} = \frac{\ln 2}{5730 \text{ yr}} \approx 0.000121 \text{ yr}^{-1}.$$

Assume that the amount of C-14 in a living bone is y_0. Over t years, the amount of C-14 in the fossilized bone decays to 30% of its initial value, or $0.3y_0$. Using the decay function, we have

$$0.3y_0 = y_0 e^{-0.000121t}.$$

Solving for t, the age of the bone in years is:

$$t = \frac{\ln 0.3}{-0.000121} \approx 9950.$$

Related Exercises 17–24 ◀

Pharmacokinetics Pharmacokinetics describes the processes by which drugs are assimilated by the body. The elimination of most drugs from the body may be modeled by an exponential decay function with a known half-life (alcohol is a notable exception). The simplest models assume that an entire drug dose is immediately absorbed into the blood. This assumption is a bit of an idealization; more refined mathematical models can account for the absorption process.

EXAMPLE 6 **Pharmacokinetics** An exponential decay function $y(t) = y_0 e^{-kt}$ models the amount of drug in the blood t hr after an initial dose of $y_0 = 100$ mg is administered. Assume the half-life of the drug is 16 hours.

a. Find the exponential decay function that governs the amount of drug in the blood.

b. How much time is required for the drug to reach 1% of the initial dose (1 mg)?

c. If a second 100-mg dose is given 12 hr after the first dose, how much time is required for the drug level to reach 1 mg?

Half-lives of common drugs

Penicillin	1 hr
Amoxicillin	1 hr
Nicotine	2 hr
Morphine	3 hr
Tetracycline	9 hr
Digitalis	33 hr
Phenobarbitol	2–6 days

SOLUTION

a. Knowing that the half-life is 16 hr, the rate constant is

$$k = \frac{\ln 2}{T_{1/2}} = \frac{\ln 2}{16 \text{ hr}} \approx 0.0433 \text{ hr}^{-1}.$$

Therefore, the decay function is $y(t) = 100e^{-0.0433t}$.

b. The time required for the drug to reach 1 mg is the solution of

$$100e^{-0.0433t} = 1.$$

Solving for t, we have

$$t = \frac{\ln 0.01}{-0.0433 \text{ hr}^{-1}} \approx 106 \text{ hr}.$$

It takes more than 4 days for the drug to be reduced to 1% of the initial dose.

c. Using the exponential decay function of part (a), the amount of drug in the blood after 12 hr is

$$y(12) = 100e^{-0.0433 \cdot 12} \approx 59.5 \text{ mg}.$$

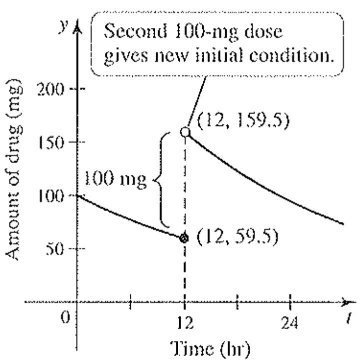

FIGURE 6.74

The second 100-mg dose given after 12 hr increases the amount of drug (assuming instantaneous absorption) to 159.5 mg. This amount becomes the new initial condition for another exponential decay process (Figure 6.74). Measuring t from the time of the second dose, the amount of drug in the blood is

$$y(t) = 159.5e^{-0.0433t}.$$

The amount of drug reaches 1 mg when

$$y(t) = 159.5e^{-0.0433t} = 1$$

which implies that

$$t = \frac{-\ln 159.5}{-0.0433 \text{ hr}^{-1}} = 117.1 \text{ hr}.$$

Approximately 117 hr after the second dose (or 129 hr after the first dose), the drug reaches 1% of the initial dose. *Related Exercises 17–24* ◀

SECTION 6.8 EXERCISES

Review Questions

1. In terms of relative growth rate, what is the defining property of exponential growth?
2. Give two pieces of information that may be used to formulate an exponential growth or decay function.
3. Explain the meaning of doubling time.
4. Explain the meaning of half-life.
5. How are the rate constant and the doubling time related?
6. How are the rate constant and the half-life related?
7. Give two examples of processes that are modeled by exponential growth.
8. Give two examples of processes that are modeled by exponential decay.

Basic Skills

9–10. Absolute and relative growth rates *Two functions f and g are given. Show that the growth rate of the linear function is constant and the relative growth rate of the exponential function is constant.*

9. $f(t) = 100 + 10.5t,\ g(t) = 100e^{t/10}$

10. $f(t) = 2200 + 400t,\ g(t) = 400 \cdot 2^{t/20}$

11–14. Designing exponential growth functions *Devise the exponential growth function that fits the given data, then answer the accompanying questions. Be sure to identify the reference point ($t = 0$) and units of time.*

11. **Population** The population of a town with a 2010 population of 90,000 grows at a rate of 2.4%/yr. In what year will the population double its initial value (to 180,000)?

12. **Population** The population of Clark County, Nevada, was 1.9 million in 2008. Assuming an annual growth rate of 4.5%/yr, what will the county population be in 2020?

13. **Rising costs** Between 2005 and 2010, the average rate of inflation was about 3%/yr (as measured by the Consumer Price Index). If a cart of groceries cost \$100 in 2005, what will it cost in 2015 assuming the rate of inflation remains constant?

14. **Cell growth** The number of cells in a tumor doubles every 6 weeks starting with 8 cells. After how many weeks does the tumor have 1500 cells?

15. **Projection sensitivity** According to the 2000 census, the U.S. population was 281 million with an estimated growth rate of 0.7%/yr.
   a. Based on these figures, find the doubling time and project the population in 2100.
   b. Suppose the actual growth rates are just 0.2 percentage points lower and higher than 0.7%/yr (0.5% and 0.9%). What are the resulting doubling times and projected 2100 population?
   c. Comment on the sensitivity of these projections to the growth rate.

16. **Oil consumption** Starting in 2010 ($t = 0$), the rate at which oil is consumed by a small country increases at a rate of 1.5%/yr, starting with an initial rate of 1.2 million barrels per year.
   a. How much oil is consumed over the course of the year 2010 (between $t = 0$ and $t = 1$)?
   b. Find the function that gives the amount of oil consumed between $t = 0$ and any future time t.
   c. After how many years will the amount of oil consumed reach 10 million barrels?

17–20. Designing exponential decay functions *Devise an exponential decay function that fits the following data; then answer the accompanying questions. Be sure to identify the reference point ($t = 0$) and units of time.*

17. **Crime rate** The homicide rate decreases at a rate of 3%/yr in a city that had 800 homicides/yr in 2010. At this rate, when will the homicide rate reach 600 homicides/yr?

18. **Drug metabolism** A drug is eliminated from the body at a rate of 15%/hr. After how many hours does the amount of drug reach 10% of the initial dose?

19. **Atmospheric pressure** The pressure of Earth's atmosphere at sea level is approximately 1000 millibars and decreases exponentially with elevation. At an elevation of 30,000 ft (approximately the altitude of Mt. Everest), the pressure is one-third of the sea-level pressure. At what elevation is the pressure half of the sea-level pressure? At what elevation is it 1% of the sea-level pressure?

20. **China's population** China's one-child policy was implemented with a goal of reducing China's population to 700 million by 2050 (from 1.2 billion in 2000). Suppose China's population declines at a rate of 0.5%/yr. Will this rate of decline be sufficient to meet the goal?

21. **Valium metabolism** The drug Valium is eliminated from the bloodstream with a half-life of 36 hr. Suppose that a patient receives an initial dose of 20 mg of Valium at midnight.
   a. How much Valium is in the patient's blood at noon the next day?
   b. When will the Valium concentration reach 10% of its initial level?

22. **Carbon dating** The half-life of C-14 is about 5730 yr.
   a. Archaeologists find a piece of cloth painted with organic dyes. Analysis of the dye in the cloth shows that only 77% of the C-14 originally in the dye remains. When was the cloth painted?
   b. A well-preserved piece of wood found at an archaeological site has 6.2% of the C-14 that it had when it was alive. Estimate when the wood was cut.

23. **Uranium dating** Uranium-238 (U-238) has a half-life of 4.5 billion yr. Geologists find a rock containing a mixture of U-238 and lead, and determine that 85% of the original U-238 remains; the other 15% has decayed into lead. How old is the rock?

24. **Radioiodine treatment** Roughly 12,000 Americans are diagnosed with thyroid cancer every year, which accounts for 1% of all cancer cases. It occurs in women three times as frequently as in men. Fortunately, thyroid cancer can be treated successfully in many cases with radioactive iodine, or I-131. This unstable form of iodine has a half-life of 8 days and is given in small doses measured in millicuries.
   a. Suppose a patient is given an initial dose of 100 millicuries. Find the function that gives the amount of I-131 in the body after $t \geq 0$ days.
   b. How long does it take for the amount of I-131 to reach 10% of the initial dose?
   c. Finding the initial dose to give a particular patient is a critical calculation. How does the time to reach 10% of the initial dose change if the initial dose is increased by 5%?

Further Explorations

25. **Explain why or why not** Determine whether the following statements are true and give an explanation or counterexample.
   a. A quantity that increases at 6%/yr obeys the growth function $y(t) = y_0e^{0.06t}$.
   b. If a quantity increases by 10%/yr, it increases by 30% over 3 yr.
   c. A quantity decreases by one-third every month. Therefore, it decreases exponentially.
   d. If the rate constant of an exponential growth function is increased, its doubling time is decreased.
   e. If a quantity increases exponentially, the time required to increase by a factor of 10 remains constant for all time.

26. **Tripling time** A quantity increases according to the exponential function $y(t) = y_0e^{kt}$. What is the tripling time for the quantity? What is the time required for the quantity to increase p-fold?

27. **Constant doubling time** Prove that the doubling time for an exponentially increasing quantity is constant for all time.

28. **Overtaking** City A has a current population of 500,000 people and grows at a rate of 3%/yr. City B has a current population of 300,000 and grows at a rate of 5%/yr.
   a. When will the cities have the same population?
   b. Suppose City C has a current population of $y_0 < 500{,}000$ and a growth rate of $p > 3\%$/yr. What is the relationship between y_0 and p such that the Cities A and C have the same population in 10 yr?

29. **A slowing race** Starting at the same time and place, Abe and Bob race, running at velocities $u(t) = 4/(t + 1)$ mi/hr and $v(t) = 4e^{-t/2}$ mi/hr, respectively, for $t \geq 0$.
   a. Who is ahead after $t = 5$ hr? After $t = 10$ hr?
   b. Find and graph the position functions of both runners. Which runner can run only a finite distance in an unlimited amount of time?

Applications

30. **Law of 70** Bankers use the law of 70, which says that if an account increases at a fixed rate of $p\%$/yr, its doubling time is approximately $70/p$. Explain why and when this statement is true.

31. **Compounded inflation** The U.S. government reports the rate of inflation (as measured by the Consumer Price Index) both monthly and annually. Suppose that, for a particular month, the *monthly* rate of inflation is reported as 0.8%. Assuming that this rate remains constant, what is the corresponding *annual* rate of inflation? Is the annual rate 12 times the monthly rate? Explain.

32. **Acceleration, velocity, position** Suppose the acceleration of an object moving along a line is given by $a(t) = -kv(t)$, where k is a positive constant and v is the object's velocity. Assume that the initial velocity and position are given by $v(0) = 10$ and $s(0) = 0$, respectively.
   a. Use $a(t) = v'(t)$ to find the velocity of the object as a function of time.
   b. Use $v(t) = s'(t)$ to find the position of the object as a function of time.
   c. Use the fact that $dv/dt = (dv/ds)(ds/dt)$ by the Chain Rule to find the velocity as a function of position.

33. **Free fall** (adapted from Putnam Exam, 1939) An object moves freely in a straight line except for air resistance proportional to its speed; this means its acceleration is $a(t) = -kv(t)$. The speed of the object decreases from 1000 ft/s to 900 ft/s over a distance of 1200 ft. Approximate the time required for this deceleration to occur. (Exercise 32 may be useful.)

34. **A running model** A model for the startup of a runner in a short race results in the velocity function $v(t) = a(1 - e^{-t/c})$, where a and c are positive constants and v has units of m/s. Source: *A Theory of Competitive Running*, Joe Keller, *Physics Today*, 26 (Sept 1973).
   a. Graph the velocity function for $a = 12$ and $c = 2$. What is the runner's maximum velocity?
   b. Using the velocity in part (a) and assuming $s(0) = 0$, find the position function $s(t)$ for $t \geq 0$.
   c. Graph the position function and estimate the time required to run 100 m.

35. **Tumor growth** Suppose the cells of a tumor are idealized as spheres each with a radius of 5 μm (micron). The number of cells has a doubling time of 35 days. Approximately how long will it take a single cell to grow into a multi-celled spherical tumor with a volume of 0.5 cm^3 (1 cm = 10,000 μm)? Assume that the tumor spheres are tightly packed.

36. **Carbon emissions from China and the United States** The burning of fossil fuels releases greenhouse gases into the atmosphere. In 1995, the United States emitted about 1.4 billion tons of carbon into the atmosphere, nearly one-fourth of the world total. China was the second largest contributor, emitting about 850 million tons of carbon. However, emissions from China were rising at a rate of about 4%/yr, while U.S. emissions were rising at about 1.3%/yr. Using these growth rates, project greenhouse gas emissions from the United States and China in 2020. Graph the projected emissions for both countries. Comment on your observations.

37. **A revenue model** The owner of a clothing store understands that the demand for shirts decreases with the price. In fact, she has developed a model that predicts that at a price of \$$x$ per shirt, she can sell $D(x) = 40e^{-x/50}$ shirts in a day. It follows that the revenue (total money taken in) in a day is $R(x) = xD(x)$ (\$$x$/shirt · $D(x)$ shirts). What price should the owner charge to maximize revenue?

Additional Exercises

38. **Geometric means** A quantity grows exponentially according to $y(t) = y_0e^{kt}$. What is the relationship between m, n, and p such that $y(p) = \sqrt{y(m)y(n)}$?

39. **Equivalent growth functions** The same exponential growth function can be written in the forms $y(t) = y_0e^{kt}$, $y(t) = y_0(1 + r)^t$, and $y(t) = y_0 2^{t/T_2}$. Derive the relationships among k, r, and T_2.

40. **General relative growth rates** Define the relative growth rate of the function f over the time interval T to be the relative change in f over an interval of length T:

$$R_T = \frac{f(t + T) - f(t)}{f(t)}.$$

Show that for the exponential function $y(t) = y_0e^{kt}$, the relative growth rate R_T is constant for any T; that is, choose any T and show that R_T is constant for all t.

QUICK CHECK ANSWERS

1. Population A grows exponentially; population B grows linearly. 3. The function $100e^{0.05t}$ increases by a factor of 1.0513, or by 5.13%, in 1 unit of time. 4. 10 yr. ◀