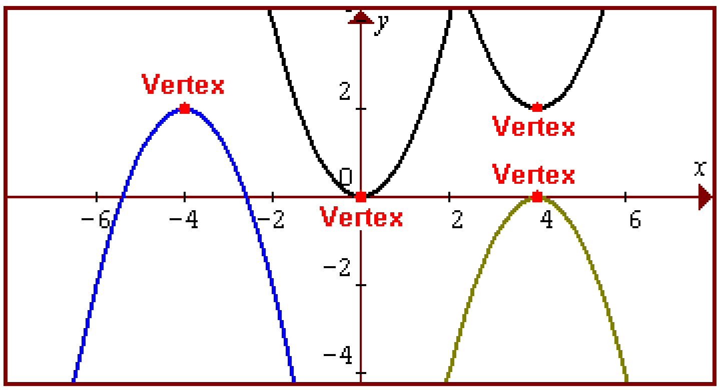
Quadratic Functions

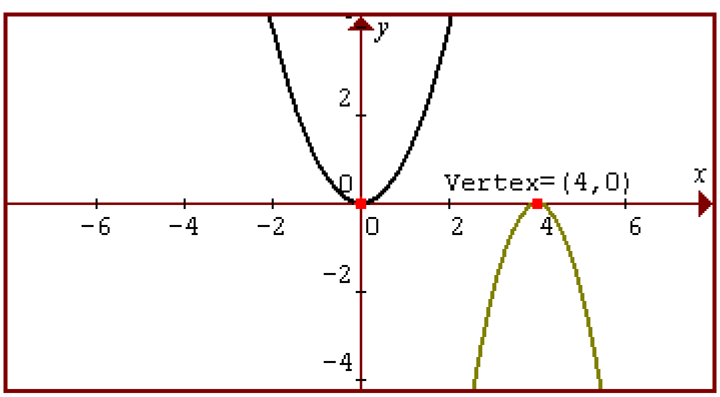
Quadratic functions are any functions that may be written in the form
$y = ax^2 + bx + c$ where a, b, and c are real coefficients and $a \neq 0$. For example, $y = 2x^2$ is a quadratic function since we have the x-squared term. $y = x^2 - 1/x + 1$ would *not* be a quadratic function because the 1/x term is equal to x^{-1} which does not fit the form. The graph of a quadratic function is called a **parabola**.

The Vertex Of The Graph Of A Quadratic Function
The vertex of the graph of a quadratic function is defined as the point where the graph changes from increasing to decreasing or changes from decreasing to increasing. Several graphs are shown below along with location of each vertex. Note that the vertex of $y=x^2$ will be (0,0).



Just Remember, The Vertex is the highest or lowest point of the graph.

Graphing $y=x^2$ and Shifts of $y=x^2$
We can obtain an equation for the graph of any quadratic function we want by applying function graph shift rules to the graph of $y=x^2$. For example, if we want a graph that is a vertical stretch of $y=x^2$ by a factor of two, is shifted right 4 units, and opens downward instead of upward, we use the formula $y = -2(x - 4)^2$. The graphs of both are shown below.



Note that the vertex has been shifted from (0,0) right 4 units. We use the vertex as our reference point for this new parabola with vertex shifted 4 units right to (4,0).

Example: Use function shift rules to predict what the graph of $y = 3(x - 4)^2 + 1$ will look like. Also, indicate the location of the vertex.

In this case, we are vertically stretching $y = x^2$ by a factor of 3 (Vertical Stretch Rule) and then shifting to the right 4 and up 1 (Function Shift Rules for Shifting Right and Up). The vertex is shifted 4 to the right and up 1 from (0,0) to (4,1).

Standard Form of a Quadratic Functions

The Standard Form of a quadratic function is

$$\mathbf{y = a(x - h)^2 + k}$$

If a quadratic function is written in this form, then it will have the same shape as $y = ax^2$ and its vertex will be shifted from (0,0) to a new location as determined by the values of h and k.

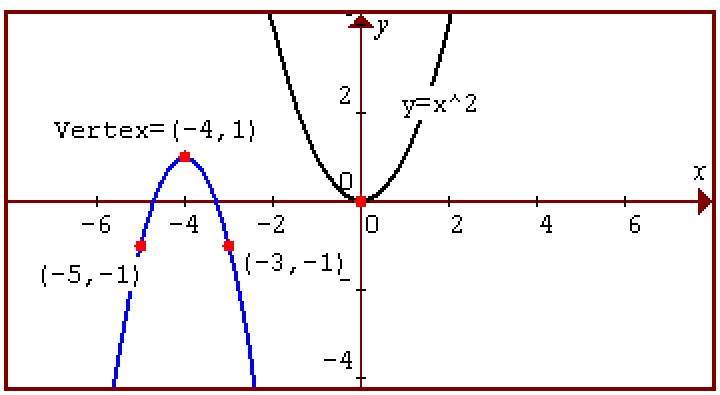
TIP: The value of "a" gives you the shape of the graph and the values of "h" and "k" tell you the location of the vertex.

Example: Graph $y = -2(x + 4)^2 + 1$ and give the value of the vertex.

This graph will have the same shape as $y = -2x^2$, which is a vertical stretch of $y = x^2$ by a factor of 2 and a reflection across the x-axis. This means we go 2 units down when moving 1 unit to the right or left of the vertex.

The vertex of this parabola is shifted from (0,0) left 4 units and up 1 unit to (-4,1). This is an application of the Left Shift Rule and the Shift Up Rule.

The graphs of $y = -2(x + 4)^2 + 1$ and $y = x^2$ are given here:



Writing An Equation In Standard Form

In order to use the Standard Form to quickly graph a quadratic, it must first be in that form. It is possible to rewrite any quadratic that is in the form $y = ax^2 + bx + c$ into standard form using the procedure below.

Procedure For Writing a Quadratic Function in the form $y = a(x - h)^2 + k$ as shown using the example $y = 2x^2 + 8x - 5$.

1. Factor out the leading coefficient a via the Distributive Property.
   $y = 2x^2 + 8x -5$ is rewritten as $y = 2(x^2 + 4x) - 5$.

2. Complete the square on $x^2 + 4x$ inside the parentheses by adding the square of half of 4, i.e. add $(4/2)^2$ or 4 inside the parentheses. Immediately subtract an equivalent amount for the same side of the equation after –5. You would subtract not 4 but $2\bullet4=8$ from the –5 term. You need to subtract the same amount that you are adding in order for the equation to not change!

   $$y = \underbrace{2(x^2 + 4x + 4)}_{\text{Add 8}} - 5 - \underbrace{2\bullet4}_{\text{Subtract 8}}$$

3. Factor your trinomial using the Distributive Property applied to this perfect square and combine the like terms. Your equation will now be in standard form.
   $\mathbf{y = 2(x + 2)^2 - 13.}$

TIP: To know that you are doing this correctly, you should be able to multiply out all terms at any point in this process and get back the equation you started with. If you get something else, you know you made a mistake!

The Shortcut Formula To The Rescue!

There is a very simple way to find the vertex of the quadratic $y = ax^2 + bx + c$.

The vertex of $y = ax^2 + bx + c$ will be at $x = -b/(2a)$. To find the y-value of this vertex, simply substitute your x-value into your equation. Furthermore, the graph will be a shift of $y = ax^2$ i.e. it will have the same shape as $y = ax^2$. It's that easy!

Example: Use the Shortcut Formula to find the vertex of $y = 2x^2 + 8x - 5$.
Then graph $y = \mathbf{2}x^2 + \mathbf{8}x - 5$.
Substitute **a** = **2** and **b** = **8** into $x= -\mathbf{b}/(2\bullet\mathbf{a})$ to get
$x = -\mathbf{8}/(2\bullet\mathbf{2}) = -8/4 = -2$.

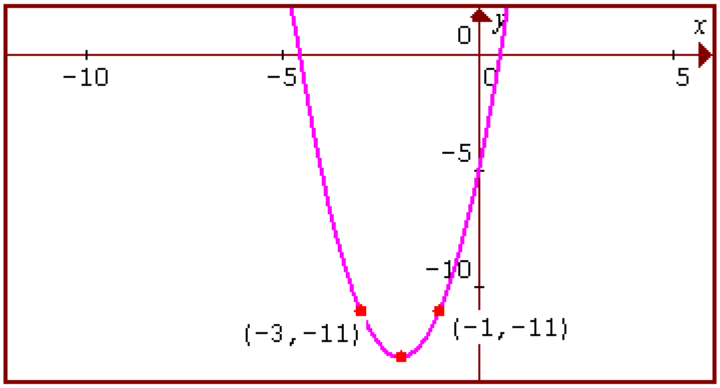
Substitute $x = -2$ into $y = 2x^2 + 8x - 5$ to get
$y = 2(-2)^2 + 8(-2) - 5$

Simplify to get $y = -13$.

The vertex is at (-2,-13). Also, we know that the graph will be a shift of $y=ax^2$, which in this case means our **graph is a shift of $y = 2x^2$**.

CONTINUED

The vertex is at (-2,-13). Also, we know that the graph will be a shift of $y=ax^2$, which in this case means our **graph is a shift of $y = 2x^2$**. We move 1 unit right and left from the vertex and two units up to plot the points (-3,-11) and (-1,-11).



Where Does This Shortcut Formula Come From?
This formula can be shown to be valid by performing the operations of writing $y=ax^2+bx+c$ in Standard Form by completing the square as shown earlier. You are able to rewrite this as

$y = a(x + b/(2a))^2 + [c - b^2/(4a)]$ after simplifying

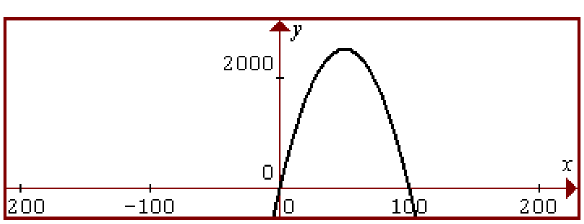
So the vertex will be at $x= –b/(2a)$, $y = c – b^2/(4a)$, and the shape will be the same as $y=ax^2$. **The operations showing this are given at the end of this lesson.**

Applications of Quadratic Functions
The vertex of a graph, depending on whether the graph opens up or down, represents the minimum or maximum value of the function. So if we are given a function and asked to find the value of our independent variable (x) that results in the maximum or minimum value of our dependent value (y), we simply need to find the value of x at the vertex.

Example: A rectangular region has an area given by $A = x(100 – x)$ sq. units where x is the width and 100-x is the height. What height and width result in the maximum area?

Here we are given the quadratic function $A = x(100 – x)$. We can use the Distributive Property to multiply this out and write as

$A = 100x – x^2$, which is equivalent to $y = -x^2 + 100x$. The vertex will be at $x = -100/(2 \bullet -1) = 50$. This means the highest value of A occurs at x = 50. So the width is x=50 units and the height is also 100-x = 50 units. See the graph below.



Verification of the Shortcut Formula

Show that the vertex of $y = ax^2 + bx + c$ is $x = -b/(2a)$ and the graph will be a shift of $y = ax^2$ and thus have the same shape as $y = ax^2$.

1. Factor out the leading coefficient a via the Distributive Property.

   $y = \mathbf{a}x^2 + bx + c$ is rewritten as $y = \mathbf{a}(x^2 + b/a\ x) + c$

2. Complete the square on the quantity inside the parentheses. You would take ½ of the quantity b/a to get b/(2a). Square this to give you b2/(4ac). Add $b^2/(4a^2)$ inside the parentheses. Subtract an equivalent amount from c on the same side of the equation. Note that you don't simply subtract $b^2/(4ac)$, but rather $a \bullet b^2/(4ac)$. You should get

   $y = \mathbf{a}(x^2 + b/a\ x + b^2/(4a^2)\ ) + c - a \bullet b^2/(4a^2)$

3. Factor the trinomial. You get

   $y = \mathbf{a}(x + b/(2a))^2 + [c - a \bullet b^2/(4a^2)]$ or
   $\mathbf{y = a(x + b/(2a))^2 + [c - b^2/(4a)]}$ after simplifying

   So the vertex will be at x= – **b/(2a)** since it is shifted left **b/(2a)** units from (0,0) according to the Left Shift Rule. As an added result, we find that the y-value of the vertex will be $y = \mathbf{c - b^2/(4a)}$ according to the Up Shift Rule. Finally, the shape will be the same as $y = \mathbf{a}x^2$.