11.4 Integration by Parts

Chain Rule ⇝ Substitution Rule
Product Rule ⇝ Integration by Parts.

$$\int \left(\overset{g}{e^x} \cdot \overset{f'}{2x} + \overset{f}{x^2} \cdot \overset{g'}{e^x} \right) dx = e^x x^2 + C$$

PRODUCT RULE:

$$\frac{d}{dx}\left[f(x)g(x)\right] = f(x)g'(x) + f'(x)g(x)$$

So:

$$\int \left(f(x)g'(x) + f'(x)g(x) \right) dx = f(x)g(x)$$

$$\int f(x)g'(x)\,dx + \cancel{\int f'(x)g(x)\,dx} = f(x)g(x)$$

$$- \cancel{\int f'(x)g(x)\,dx} \qquad - \int f'(x)g(x)\,dx$$

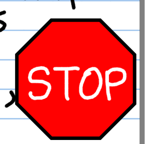
$$\boxed{\int f(x)\cdot g'(x)\,dx = f(x)g(x) - \int f'(x)g(x)\,dx}$$

Ex:
$$\int x e^x\,dx = \overset{f}{x} \cdot \overset{g}{e^x} - \int 1 \cdot e^x\,dx$$

$f(x) = x$ $\quad g'(x) = e^x$
$f'(x) = 1$ $\quad g(x) = e^x$

$$= \boxed{x e^x - e^x + C}$$

With integration by parts, our goal is to integrate a SIMPLER function. If your choice of $f(x)$ and $g'(x)$ is making things more complicated,



and try something different!

In differential notation, Integration by Parts looks like:

$$\int \overset{f \cdot g'}{u \cdot dv} = \overset{f\,g}{uv} - \int \overset{g \cdot f'}{v \cdot du}$$

Ex 2 $\int x \cos 3x \, dx$

$u = x$ $\quad dv = \cos 3x \, dx$

$\downarrow$ deriv. $\quad \downarrow \int$

$\frac{du}{dx} = 1 \quad V = \frac{1}{3}\sin 3x$

$du = dx$

so:

$$\int u \, dv = uv - \int v \cdot du$$

$$\int \overset{u}{x} \cdot \overbrace{\cos 3x \, dx}^{dv} = x \cdot \frac{1}{3}\sin 3x - \int \frac{1}{3}\sin 3x \cdot dx$$

$$= \frac{1}{3}x\sin 3x - \frac{1}{3}\left(-\frac{1}{3}\cos 3x\right) + C$$

$$= \frac{1}{3}x\sin 3x + \frac{1}{9}\cos 3x + C$$

$$\int x^2 \ln x \, dx$$

$u = \ln x \qquad dV = x^2$

$du = \frac{1}{x} dx \qquad V = \frac{1}{3}x^3$

$$\int x^2 \ln x \, dx = \ln x \cdot \frac{1}{3}x^3 - \int \frac{1}{3}x^3 \cdot \frac{1}{x} dx$$

$$= \frac{1}{3}x^3 \ln x - \frac{1}{3}\int x^2 dx$$

$$= \frac{1}{3}x^3 \ln x - \frac{1}{3}\left(\frac{1}{3}x^3\right) + C$$

$$= \boxed{\frac{1}{3}x^3 \ln x - \frac{1}{9}x^3 + C}$$

TRY THIS: (Hi James! Having fun figuring this out, yet?!)

BTW:
$\int \tan x \, dx$
$= \ln|\sec x| + C$

$$\int t \cdot \sec^2 t \, dt =$$

$u = t \qquad dV = \sec^2 t \, dt$

$du = dt \qquad V = \tan t$

$$\int t \sec^2 t \, dt = t \tan t - \int \tan t \, dt$$

$$= t \tan t - \ln|\sec x| + C$$

$$\int \overset{u}{x^2}\ \overset{dV}{\sin 3x\,dx}$$

$u = x^2$ $\quad dV = \sin 3x\,dx$

$du = 2x\,dx$ $\quad V = -\frac{1}{3}\cos 3x$

$$u \cdot v - \int v \cdot du$$

$$= x^2 \cdot \frac{-1}{3}\cos 3x - \int -\frac{1}{3}\cos 3x \cdot 2x\,dx$$

$$= -\frac{1}{3}x^2\cos 3x + \frac{2}{3}\int x\cos 3x\,dx$$

new $u = x$ $\quad$ new $dV = \cos 3x\,dx$

$du = dx$ $\quad V = \frac{1}{3}\sin 3x$

$$= -\frac{1}{3}x^2\cos 3x + \frac{2}{3}\left[x \cdot \frac{1}{3}\sin 3x - \int \frac{1}{3}\sin 3x\,\underset{1}{dx}\right]$$

$$= -\frac{1}{3}x^2\cos 3x + \frac{2}{9}x\sin 3x - \frac{2}{9}\int \sin 3x\,dx$$

$$= -\frac{1}{3}x^2\cos 3x + \frac{2}{9}x\sin 3x - \frac{2}{9}\left(-\frac{1}{3}\cos 3x\right) + \frac{2}{3}C$$

$$= -\frac{1}{3}x^2\cos 3x + \frac{2}{9}x\sin 3x + \frac{2}{27}\cos 3x + K$$

$$\int \overset{u\,\cdot\,dv}{\ln x\, dx} = x\ln x - \int x\cdot\frac{1}{x}\,dx$$

$u = \ln x \qquad dv = dx$

$du = \frac{1}{x}dx \qquad v = x$

$$= x\ln x - \int dx$$

$$= x\ln x - x + C$$

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$$\int \ln x\, dx = x\ln x - x + C$$

$$\text{or } x(\ln x - 1) + C$$

$$\int \tan x\, dx = \ln|\sec x| + C$$

$$\int \tan x \, dx = \int \frac{\sin x}{\cos x} \, dx$$

using Substitution

If $u = \cos x$

$du = -\sin x \, dx$

so :

$$\int \frac{\sin x}{\cos x} \, dx = \int \frac{-du}{u} = -\ln|u| + C$$

$$= \ln|u^{-1}| + C$$

$$= \ln|(\cos x)^{-1}| + C$$

$$= \ln|\sec x| + C$$