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VENTURE CAPITAL INVESTMENT: THE ROLE OF PREDATOR–PREY DYNAMICS WITH LEARNING BY DOING

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This paper suggests that endogenous dynamics of the 'predator–prey' type can provide a contributing explanation for both high-venture capital concentration by industry and 'boom and bust' industry-level investment dynamics. We propose a model based on the idea that venture capitalists favor industries where they have significant experience and industries with a large pool of good investment opportunities. However, investment 'uses up' opportunities and therefore tends to deplete the pool of unexploited opportunities. The resulting industry-level interactive dynamics naturally give rise to venture capital investment cycles similar to observed patterns.

Keywords: Venture capital; Dynamics; Predator–prey; Concentration; Learning

JEL Classification: G24; M21

1 INTRODUCTION

One striking fact about venture capital is its extraordinary concentration by industry. For example, in 2006, ∼20% of all venture capital invested in the United States went into the software industry despite the fact that software, very broadly defined, accounted for under 3% of GDP.[1] Similarly, biotechnology and telecommunications also receive disproportionately large shares of venture capital. Other industries such as finance, general manufacturing, and retailing receive correspondingly small amounts of venture capital.

At first glance, this concentration of venture capital by industry might seem unsurprising and readily explained. What is more surprising and harder to explain, however, is the dramatic variation over time in the importance of different sectors. For example, over the 1995–2007 period, the share of software in venture capital investment varied between 12% (in the first quarter of 1995) and over 28% (in the fourth quarter of 2001). Biotechnology and medical devices went through an even stronger cycle, with a share of ∼25% in the first quarter of 1995, dropping to under 5% in the third quarter of 2000, and recovering to over 25% by mid-2002. Relatively little attention has been paid to this striking time series variation in concentration.

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[1] Venture capital data reported here is from PWCMoneytree, available online at www.pwcmoneytree.com (accessed May 2007). GDP data is available online from the US Department of Commerce Bureau of Economic Analysis at www.bea.doc.gov and from the US census at www.census.gov

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It is natural to explain both the concentration of venture capital and its variation over time on the basis of various factors that affect differential profit opportunities by industry but are exogenous to specific venture capitalists (VCs). Such factors might include the rate of scientific progress in different areas, the nature of exit opportunities, the structure of regulation and taxation, etc. The importance of such factors has been the focus of significant attention in the venture capital literature. In this paper, however, we take a different approach. Specifically, we ask whether the 'boom and bust' aspect of industry-level venture capital investment shares might be generated in part by learning-by-doing on the part of VCs. If so, we could think of such cyclical dynamics as being endogenously generated within the venture capital sector rather than being the product of cyclical changes in exogenous factors.

Accordingly, the primary objective of this paper is to offer a formal theory of venture capital investment based on VC experience or 'learning-by-doing'. We seek to demonstrate that such a theory is consistent with both the persistent concentration of venture capital in a few industries and the cyclical pattern of venture capital shares by industry. A second objective is to offer a partial explanation of the dramatic boom and bust in venture capital investment surrounding the turn of the 21st century.[2] While often attributed to the rise and fall of high technology stock prices over the same period, the venture capital investment pattern could be explained in part by the endogenous dynamics that underlie industry-level cycles in VC investment.

In focusing on endogenous industry-level investment dynamics based on learning by VCs, we in no way wish to ignore or discount important exogenous determinants of venture capital investments of the type mentioned earlier. Our contribution is to suggest that, in addition to these (and other) factors that are largely exogenous to individual venture capital firms, it is possible that endogenous dynamics might also play an important role.

In addition to learning, we also focus on two other considerations. One consideration is a potential countervailing force to VC learning. Specifically, when a specific area receives a large amount of venture capital, meaning that many promising opportunities are taken leaving fewer good opportunities for other VCs. Thus high levels of venture capital investment in a particular area tend to deplete the set of unexploited opportunities. The other factor we consider is the natural growth of investment opportunities. We expect venture capital investment in areas where new scientific and technological developments generate new investment opportunities at a rapid rate. Even without exogenous changes or random shocks to other variables, such a dynamic system might well generate the 'boom and bust' dynamics and the concentration by industry, which are observed in venture capital investment.

Learning-by-doing, exhaustion of investment opportunities, and natural growth of investment opportunities are not unique to venture capital. Most financial markets are affected by such considerations. However, we suggest that these factors are particularly significant in venture capital. This applies especially to learning-by-doing.

VCs are often involved with the management of their client firms. VCs help startups find Q1 alternative equity financing (Gorman and Sahlman, 1989; Erhlich *et al.*, 1994), key management personnel (Hellmann and Puri, 2002), and candidates for licensing or acquisitions (Gans *et al.*, 2002). They also reduce the time required to bring products to market (Hellmann and Puri Q2 2000), increase the likelihood of an IPO (Hsu, 2006a), provide certification (Megginson and Weiss, 1991; Hsu, 2004), and improve governance structures (Hochberg, 2005). We expect that the impact of these value-added services on firm profits and ultimately on VC profits depends in part on VC experience. It therefore seems plausible that experience should have a positive impact on VC investment and performance.

[2] As reported in PWCMoneytree (May, 2007), available online at www.pwcmoneytree.com, annual venture capital investment in the United States grew by a factor of 5 between 1995 and 2000 and then fell by ∼80% to the 2003 trough before a slow recovery began.

There is a significant literature documenting the importance of such experience effects,[3] including accounts of interviews with VCs. When queried about the highly concentrated pattern of venture capital, respondents commonly suggest that VCs favor certain industries because of their prior experience and hence high value-added in those areas. A venture capital manager quoted in Wüstenhagen and Teppo (2005, p. 28) states that '[p]eople tend to invest in technologies they know ... [a]nd most of the funds have people that come from ... IT, telecom or life sciences'. See also Cumming (2005) for a discussion of learning by VCs. Q3

This experience effect is consistent with recent empirical work, documenting the positive impact of VCs' industry experience on investment (Sorenson and Stuart, 2001; Gompers *et al.*, 2005) and the positive relationship between VC experience and performance in venture capital investment (Sorenson and Stuart, 2001; Gompers *et al.*, 2005; Kaplan and Schoar, 2005; Sorensen, 2005; Hochberg *et al.*, 2006; Kaplan *et al.*, 2006). As documented in Lindsey (2002), VCs also take advantage of their experience by facilitating collaborations among their portfolio firms, who then benefit from a 'keiretsu' effect. Q4,Q5

Our paper is also related to a recent strand in the literature that examines the key factors affecting venture capital activity.[4] Such factors might include VC efficiency in selecting and monitoring investments relative to other types of investors (Amit *et al.*, 1998), the availability of IPOs as exit options for VCs (Jeng and Wells, 2000), firms' ability to appropriate (part of) the marginal social product of their innovations (Gans and Stern, 2003), the liquidity of stock markets and the stock of human capital in the economy (Schertler, 2003), returns to venture capital investments (Inderst and Müller, 2004), and the legal environment (Megginson, 2004).

Overall, this literature suggests that cross-industry and temporal variations in VC activity would normally result from variations in exogenous factors of the type described in the previous paragraph. For example, Amit *et al.* (1998) provided a static agency model in which, if extended to a dynamic framework, variation in venture capital investment would most naturally arise from differences in the degree of information asymmetry between the entrepreneur and the VC. In Gans and Stern (2003), venture capital fluctuations were explained by the degree of appropriability of inventions and other exogenous factors. Even in dynamic models such as in Inderst and Müller (2004), an increase in returns to venture capital yielded a one-time increase in long-term venture capital activity through entry of firms in the industry, but not to multiple investment cycles.[5]

Notwithstanding the literature noting the importance of VC experience and the literature addressing VC investment patterns, we believe that this paper offers a unique and interesting contribution. Specifically, our contribution is to propose a formal model of cyclical dynamics in which cross-industry and temporal variations in venture capital activity emerge endogenously as a result of learning-by-doing, the impact of investment on unexploited opportunities, and the natural growth of the opportunity pool, rather than from successive exogenous perturbations. Although we are certainly not the first researchers to emphasize the importance of learning-by-doing for VCs, we believe that we are the first to demonstrate that experience effects might naturally lead to industry-level venture capital investment cycles.

A central component of this contribution arises from our application of the traditional Lotka–Volterra biological predator–prey model to venture capital investment. This model was

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[3] Entrepreneurs' experience also plays a role in startups. See, e.g. Hsu (2006b), who found that founder experience increases venture valuation.

[4] There is a large general literature on venture capital that we not review here. Useful background is covered in Gompers and Lerner (1999) and in Wright *et al.* (2003). An influential analysis of VC contracts is in Kaplan and Strömberg (2003). See Bottazzi *et al.* (2004) for analysis of the European venture capital industry and Dosi (1988) for an overview of the effects of innovation. Q6

[5] Similarly in the herding literature, a single, exogenous, positive (negative) shock leads to a "cascading" increase (decrease) in activity/investment, but not to cyclical behavior. See for example Banerjee (1992), Bikhchandani *et al.* (1992), and Scharfstein and Stein (1990).

originally used to explain otherwise very puzzling cycles in wildlife populations. We use the model to explain otherwise puzzling cycles in venture capital investment concentration. The structure of predator–prey models is clearly presented in standard textbooks on differential equations, such as Boyce and de Prima (2005). Such models have been applied to other areas of economics, including renewable natural resources (Brander and Taylor, 1998; Pezzey and Anderies, 2003), the economics of the environment and optimal harvesting rates (Hoekstra and van den Bergh, 2005), and labor economics and union bargaining (Kremer and Olken, 2006). Our model is novel in expanding the application of the Lotka–Volterra structure to a new area, namely venture capital investment. It generates interesting and plausible dynamic behavior, and we hope that it offers a new and useful perspective on entrepreneurial finance more broadly. Specifically, we suggest that the observed cycle in VC investment concentration by industry might be viewed as arising from the transitional dynamics associated with the Lotka–Volterra model.

Section 2 provides a brief description of the empirical regularities that we wish to explain. Section 3 sets out our formal model, and Section 4 provides an analysis of the steady-state and transitional dynamic properties of the model. Section 5 is devoted to simulation results showing that the model is consistent with the major stylized facts regarding venture capital dynamics. Section 6 contains concluding remarks. Appendix A provides a detailed discussion of the data underlying our reported stylized facts. Appendix B provides a formal statement and proof of some of the more technical results in the paper.

2 DATA DESCRIPTION

We highlight three important empirical regularities regarding venture capital. First, Figure 1 shows the evolution of quarterly venture capital investment in the United States from the first quarter of 1995 through the first quarter of 2007. The figure also shows total investment and investment in the three major recipient industries of venture capital (biotechnology and medical devices, software, and telecommunications and networking).

Figure 1 shows a dramatic boom and bust pattern. Venture capital investment peaked in the first quarter of 2000 at about $28.4 billion. By the first quarter of 2003, venture capital

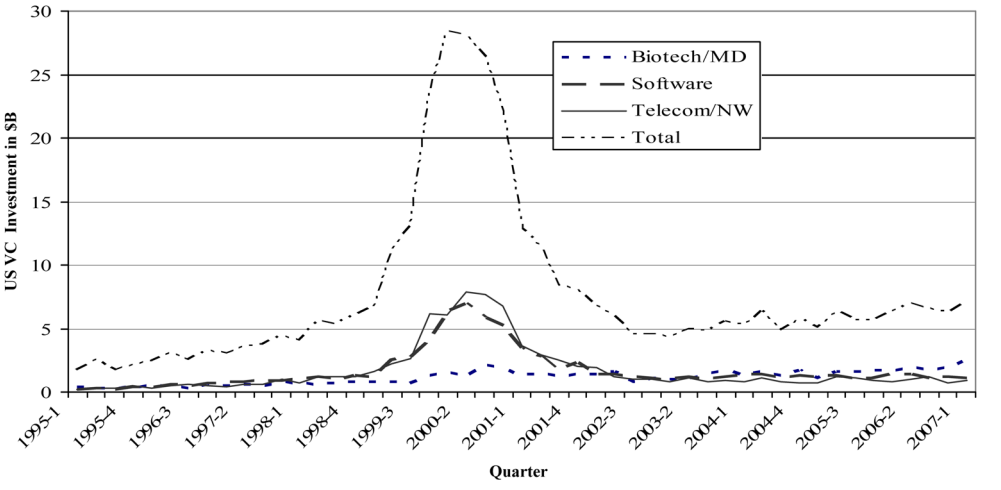

FIGURE 1 Quarterly VC investment by industry and total: Q1, 1995–2007. Source: PWCMoneytree.com, accessed May 2007.

TABLE I Venture Capital Shares and GDP shares by industry (2002) (VC investment and GDP are in $billions).

Industry	*VC invest*	*VC share (%)*	*GDP*	*GDP share (%)*	*VC share to GDP share*
Biotechnology and medical equipments	3.18	14.6	227	2.2	6.73
Business products and services	0.52	2.4	986	9.4	0.25
Computers and peripherals	0.45	2.1	99	0.9	2.19
Consumer products and services	0.24	1.1	383	3.7	0.30
Electronics/instrumentation	0.30	1.4	116	1.1	1.24
Financial services	0.35	1.6	2126	20.3	0.08
Healthcare services	2.22	10.2	702	6.7	1.52
it services	1.06	4.9	43	0.4	11.88
Industrial/energy	0.72	3.3	1860	17.7	0.19
Media and entertainment	0.74	3.4	119	1.1	2.98
Retailing/distribution	0.16	0.7	1333	12.7	0.06
Semiconductors	1.55	7.1	61	0.6	12.22
Software	5.21	23.9	313	3.0	7.99
Telecommunications and networking	5.10	23.4	321	3.1	7.63
Undisclosed/other	0.02	0.1	1792	17.1	0.01
Totals	21.84	100.0	10481.00	100.0	1.00

Sources: Venture capital data is from PWC Moneytree. GDP data is obtained from 2002 census-based GDP information available at www.census.gov. Details are in Appendix A.

investment had fallen to about $4.3 billion – a drop of ∼85% and similar to the $4.1 billion reported in the first quarter of 1998. The number of investments or 'deals' (not shown) did not show quite as dramatic a rise and fall, implying that the peak period for venture capital investment was characterized by higher average deal sizes as well as more deals. Even so, the pattern of deals displays a very marked boom and bust in itself.

The three specific industries shown in Figure 1 together received a large share of total venture capital investment, ∼60% over the full period shown. This reflects the second major empirical regularity – the striking concentration of venture capital in a few industries. Table I illustrates this fact by showing the share of venture capital investment received by each sector in 2002[6] and the corresponding share of GDP by sector.

Venture capital intensity is very high for biotechnology (including medical equipment), IT services, semiconductors, software, and telecommunications (including networking). For these industries, the ratio of the venture capital share to the GDP share exceeds 6. Major traditional industries, on the other hand, received much less than their GDP-based 'share' of venture capital. Admittedly, this partially reflects the fact that established industries have more access to other forms of finance, as is consistent with their risk and information characteristics. Even so, the high concentration of venture capital in certain industries is noteworthy.

What is much less well known than the overall boom and bust in venture capital investment – and this is our third important empirical regularity – is that the relative importance of the different sectors as recipients of venture capital exhibits a marked cyclical pattern. It is hard to see the relative cycle across industries in Figure 1, given the dominant visual effect of the aggregate cycle. However, the relative cycle is evident in Figure 2, which shows the share of venture capital investment received by the three large recipient industries.

Figure 2 shows a striking pattern of cyclical relative importance for the three venture capital recipient industries shown. For example, the biotechnology share in 2000 was low, falling under 5%, then it rose sharply in the following two years, and ultimately reached 36% in Q1, 2007. Figure 2 provides a different perspective from Figure 1. As can be seen in Figure 1, despite

[6] We use 2002, as it is the most recent year for which sufficiently disaggregated GDP data is available. Another disaggregated GDP census will be done for the 2007 year.

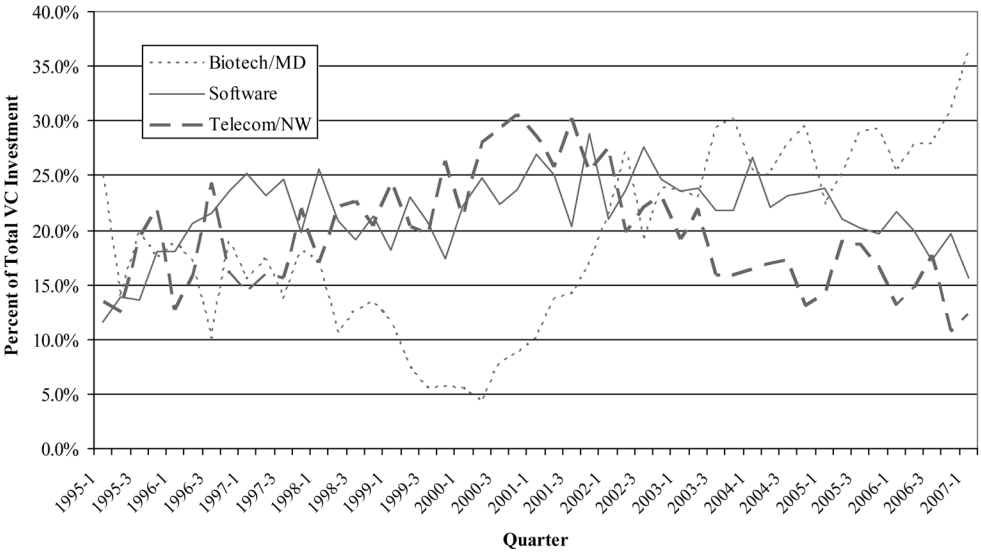

FIGURE 2 Evolution of industry-level VC investment shares Q1, 1995–2007. Source: PWCMoneytree.com, accessed May 2007

its low VC investment share, biotechnology's absolute level of VC investment reached a peak in the latter part of 2000 that was not reached again until 2006. Correcting for the aggregate cycle, as Figure 2 does, is an important part of understanding VC investment dynamics.

The three stylized facts illustrated above may be attributable to exogenous factors such as, for example, the boom and bust cycle that characterized high technology stock prices over the past decade. However, in the model that follows, we suggest that these empirical regularities might arise even without changes in exogenous factors, as they may also result in part from the endogenous dynamics of industry-level venture capital investment.

3 THE MODEL

The starting point of our formal analysis is the decision problem faced by VCs. A representative VC is assumed to be risk neutral and is assumed to maximize the expected present value of profits associated with current investment decisions. At the beginning of any given time period, the choice to be made by the VC is the amount of investment, v_i, to be allocated to each industry i during that period. A VC might specialize fully in one industry but, more generally, might have investments in a range of industries. We write this aggregate expected present value for a given VC as

$$\pi = \sum \pi_i(v_i; x_i, P_i, Z_i), \tag{1}$$

where π_i is the VC's (expected present value of) profit from its investment, v_i, in industry i. This industry-specific profit, π_i, depends in part on the industry-specific investment level, v_i. It also depends in part on the VC's prior experience in this industry, denoted x_i, and on the pool of investment projects available in industry i, denoted P_i.

Variable Z_i represents a vector of other exogenous influence on profits, such as interest rates, demand conditions, etc. The VC's decision problem is implicitly a dynamic problem, as the decision of how much to invest must be repeated each period. Maximization of Eq. (1) is therefore repeated each period. For our purposes, it is sufficient to characterize formally

the decision for one particular time period, recognizing that the VC looks forward and determines an expected present value associated with current decisions. Maximization of Eq. (1) with respect to the industry-specific investments $v_1, v_2, \ldots, v_n$ implies that the firm's optimal investment in any one industry will depend on the predetermined or exogenous variables x_i and P_i. Thus, we can write

$$v_i = v_i(x_i, P_i, Z_i). \tag{2}$$

Our assumption of risk neutrality implies that there is no risk aversion motive that would, in itself, induce diversification across industries. In addition, we assume that experience in industry i improves profitability only in industry i; it does not improve profitability in other industries. Thus, there is no 'economy of scope' that would in itself give rise to cross-industry diversification. A VC might still have a diversified portfolio, but this would be a reflection of the VC's opportunities, cost of raising capital, experience, and value-added capability in each industry taken independently. Abstracting from risk aversion and economies of scope across industries is a significant abstraction, but it allows us to focus as clearly as possible on the reality that high levels of specialization are important in venture capital finance.

To some extent, the degree of specialization depends on whether the unit of analysis is the venture capital firm or the venture capital fund. Typically, a given venture capital firm might have several funds, sometimes in different industries or sectors. Thus funds are more specialized than firms. Typically, each fund within a firm is managed independently with particular dedicated fund managers. For our analysis, it is probably appropriate to think of the unit of analysis as the fund rather than the firm but this is not essential and we do not refer to this distinction further.

For the United States, Hochberg *et al.* (2005) reported that 46.2% of venture capital funds focus on computer-related companies, 18.9% on 'non-high-technology', 15.5% on communications and media, and 9.2% on medical, health, and life sciences. Venture capital firms like Hummer Windblad and Tallwood – which focus exclusively on software and semiconductors, respectively – are extremely specialized, whereas other VC firms have a more diversified approach. In any case, we abstract from portfolio diversification issues. Incorporating such effects would not offset the issues we emphasize here, but it would cloud the analysis by introducing more algebraic complication.

Our focus on learning is implemented by assuming that a given project in industry i will be more profitable to the VC, other things equal, if the VC has more experience in that industry. VCs are therefore more inclined to invest in industry i if they have more experience with that industry: $\mathrm{d}v_i/\mathrm{d}x_i > 0$. At the time v_i is determined, x_i is predetermined or exogenous to the current decision.

Another very important consideration relates to the supply of available projects in the potential investment pool. The pool of possible projects is driven in large part by underlying scientific progress. At any one time, we can imagine that there is a pool of not yet fully exploited technologies applicable to a particular industry. The larger the pool, the better chance the firm has of finding profitable projects to invest in. VCs are more inclined to invest if the pool of investments is larger. Accordingly, $\mathrm{d}v_i/\mathrm{d}P_i > 0$.

The data that we discussed in Section 2 relates to industry-level investment dynamics, not to investment dynamics at the level of the individual VC fund. Accordingly, we now focus on aggregate investment in a given industry, which is the sum of investments of the individual funds. We denote this aggregate investment as V_i and the aggregate experience as X_i. We assume that this aggregation allows investment in industry i to be written as

$$V_i = V_i(X_i, P_{i,}, Z_i), \tag{3}$$

where $\mathrm{d}V_i/\mathrm{d}X_i > 0$ and $\mathrm{d}V_i/\mathrm{d}P_i > 0$.

Equation (3) has been derived on the assumption that, at any given time, risk-neutral VCs seek to maximize the present value of their investments. This allows VCs to anticipate the effect of current experience on future profits. However, such an investment rule could arise from myopic maximization of profits over some relatively short time horizon, or it could arise from some objective other than strict profit maximization. The basic point is that a large range of plausible descriptions of VC behavior would give rise to the result that industry-level investment would be increasing in prior experience in industry i and in the pool of available projects. It is not necessary for us to take a firm position of whether VCs undertake full dynamic optimization or whether they adopt a more short-run horizon.[7] It is challenging (albeit not impossible) to think of any reasonable decision process that would not have the basic properties we assume. Therefore, we view our approach as relatively general as far as assumptions about VC motivation and foresight are concerned.

It is helpful to assume a specific functional form for Eq. (3). This form is intended to be illustrative rather than universal. Other plausible functional forms would have similar properties. Furthermore, we now focus on one industry at a time and therefore drop the subscript i, so as to keep the notation as simple as possible. Also, although noting that exogenous factors captured in Z (such as interest rates, etc.) are relevant to profitability, they are not the focus of our analysis. We therefore drop Z from further consideration; we suggest a Cobb–Douglas form for investment. Accordingly, investment V in industry i is given by

$$V = \alpha X^{\beta} P. \tag{4}$$

Equation (4) indicates that investment opportunities are essential if investment is to occur. If there are no opportunities (i.e. if $P = 0$), then investment is zero, which is a desirable property.[8] There are constant returns to opportunities in that if opportunities double, so does investment. However, there are decreasing returns to experience as reflected by β, which we assume is <1 but strictly positive. The stock of relevant experience depreciates over time, and at some point, previous experience with earlier technology might become irrelevant altogether. The depreciation rate is denoted δ. The equation of motion for the stock of relevant experience is therefore

$$\begin{aligned} \frac{\mathrm{d}X}{\mathrm{d}t} &= V - \delta X \\ &= \alpha X^{\beta} P - \delta X. \end{aligned} \tag{5}$$

The stock of exploitable opportunities, P, also evolves over time. As VCs invest in a specific venture, that particular opportunity is taken out of the pool of unexploited opportunities.[9] We can think of VCs as 'predators' and investment opportunities as 'prey'. This predator–prey effect implies that the pool of opportunities is decreasing in V. However, this pool is also subject to natural proportional growth at rate ρ, reflecting underlying scientific and technological progress. The combination of depletion due to current investment and natural

[7] Klepper (1996) analyzed a related context in which firms maximize short-term profits rather than undertake full dynamic maximization. A useful argument in favor of this assumption is provided on p. 580.

[8] Investment is also zero if experience is zero, which we view as undesirable. However, the model is essentially the same, with more complicated algebra, if we use $(1 + X)$ rather than just X in Eq. (4), which allows for positive investment even if venture capitalists have no experience in this industry.

[9] We recognize that venture-funded firms often have several rounds of financing and more than one VC investor at any given round. Thus one firm has several 'opportunities' associated with it. Each investment 'uses up' one of these opportunities.

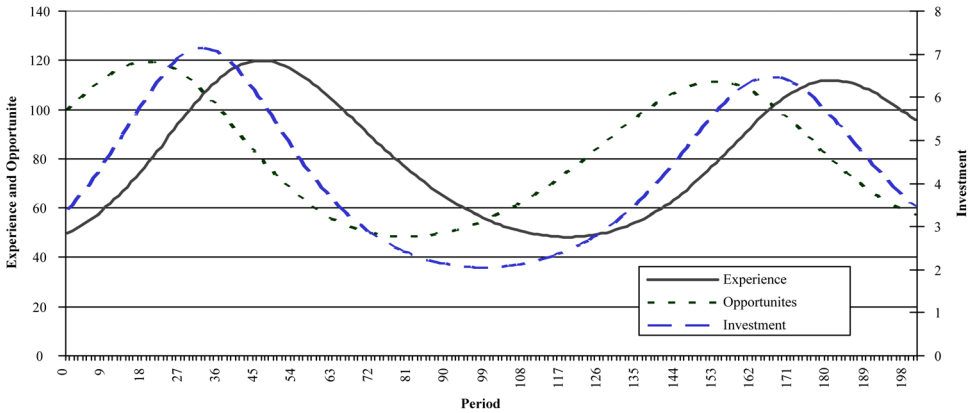

FIGURE 3 Venture capital dynamics at the industry level.

growth at proportional rate ρ leads to the following equation of motion for the pool of resources.

$$\begin{aligned} \frac{\mathrm{d}P}{\mathrm{d}t} &= \rho P - V \\ &= \rho P - \alpha X^{\beta} P \end{aligned} \tag{6}$$

The stock of experience and the pool of investment opportunities can therefore be thought of as two state variables in a two-equation dynamic system with 'equations of motion' given by Eqs. (5) and (6).

Even this simplified representation of venture capital investment can generate interesting dynamic behavior that is consistent with observed patterns. We analyze this system formally in the next section, but we first illustrate the evolution of the system for plausible parameter values. Specifically, if we normalize the two state variables (experience and opportunities) to start at 50 and 100, respectively, if parameter $\alpha = 0.001$, $\beta = 0.9$, if opportunity growth, ρ, is 5% per period, and if depreciation, δ, is also 5% per period, the pattern shown in Figure 3 arises.

As can be seen from Figure 3, this set of parameter values generates a slightly damped pattern for the two state variables (relevant experience and available opportunities) and for investment. Given the difference in scale between investment and the state variables, we have illustrated investment on a secondary vertical axis so as to see its variation clearly.

4 ANALYTICAL PROPERTIES OF VENTURE CAPITAL DYNAMICS

4.1 Steady-State Analysis

In this section, we undertake a formal analysis of the dynamic system set out in Section 3. It is useful to restate the two equations of motion given by Eqs. (5) and (6):

$$\frac{\mathrm{d}X}{\mathrm{d}t} = \alpha X^{\beta} P - \delta X \tag{5$'$}$$

$$\frac{\mathrm{d}P}{\mathrm{d}t} = \rho P - \alpha X^{\beta} P. \tag{6$'$}$$

The parameters are α, β, δ, and ρ. All are strictly positive, and β is taken to be <1. State variables, X and P, are bounded from below by 0. The formal conditions we impose are as follows.

(C1) α, β, δ, and $\rho > 0$.
(C2) $\beta < 1$.
(C3) $X, P \geq 0$.

The dynamic system given by Eqs. (5′) and (6′) has two steady states in state variables X (relevant experience) and P (the pool of available opportunities). It is clear by inspection of Eqs. (5′) and (6′) that one steady state is the trivial steady state at $(X, P) = (0, 0)$. The other solution is readily obtained by the following algebraic steps:

$$\begin{aligned} \frac{\mathrm{d}P}{\mathrm{d}t} &= \rho P - \alpha X^{\beta} P = 0 \\ &\Rightarrow \rho = \alpha X^{\beta} \\ &\Rightarrow X* = \left(\frac{\rho}{\alpha}\right)^{1/\beta}. \end{aligned} \tag{7}$$

Setting $\mathrm{d}X/\mathrm{d}t = 0$ implies that $\alpha X^{\beta} P - \delta X = 0$, which in turn implies that $P = (\delta/\alpha)X^{1-\beta}$. Substituting $X^* = (\rho/\alpha)^{1/\beta}$ (from Eq. (7)) into this equation yields $P^* = (\delta/\alpha)(\rho/\alpha)^{(1-\beta)/\beta}$, or

$$P* = \left(\frac{\delta}{\rho}\right)\left(\frac{\rho}{\alpha}\right)^{1/\beta}. \tag{8}$$

It follows from condition (C1) that that X^* and P^* are both positive, so this is an interior solution.

If state variables X and P are in steady state, then Eq. (4) implies that investment V is also in steady state. From Eqs. (4), (7), and (8) (interior) steady-state investment V is given by

$$V^* = \delta\left(\frac{\rho}{\alpha}\right)^{1/\beta}. \tag{9}$$

The comparative steady-state effects of changes in the exogenous parameters on investment are obtained by differentiation of solution (9) for steady-state investment. Specifically, the interior steady-state level of investment increases if (i) experience depreciation rate δ increases, (ii) opportunity growth rate ρ increases, (iii) investment parameter α decreases, or (iv) the returns to experience parameter β decreases.

Some comparative static properties of the steady-state investment level are perhaps surprising but they can be readily understood. Note first that in the steady state, the level of venture capital investment V must exactly have equal experience depreciation, δX. Looking back at Eq. (5′), one can see that this is necessary for experience X to remain constant, a condition for steady state. Therefore, an increase in δ must lead either to a decrease in the steady-state level of experience X^*, or to an increase in the steady-state level of investment, V^*, or both. From Eq. (7), we know that X^* is independent of δ, and hence an increase in δ must be entirely compensated in steady state by an increase in venture capital investment V^*. The comparative steady-state effect of increases in ρ is very intuitive, as higher natural growth rate of opportunities gives rise to higher steady-state levels of experience opportunities. Investment is therefore also higher, as implied by Eq. (4).

The effect of changes in α is more surprising. Smaller values of α have the direct effect (from Eq. (4)) of reducing investment. However, the indirect effects operating through the new steady-state values of P and X more than offset the direct effect, implying that steady-state investment increases when α decreases. The reasoning regarding changes in β is similar.

4.2 Transitional Dynamics

We would not normally expect conditions to remain stable long enough for a non-trivial steady state to be closely approximated. We are therefore primarily interested in transitional dynamics. We have in mind a system that starts out at initial conditions other than a steady state and evolves toward the steady state until parameters change due to some exogenous shock, resulting in a new steady state and a new trajectory of approach. The main characterization of the (local) dynamic behavior of the system is reported in the following statements.

(1) Venture capital dynamics in the neighborhood of the interior steady state may exhibit either monotonic or cyclical behavior.
(2) Cyclical behavior arises if and only if

$$\delta < \frac{4\rho\beta}{(1-\beta)^2}. \tag{10}$$

(3) For both cyclical and monotonic cases the system exhibits local convergence toward the interior steady state. This steady state is therefore stable.

These statements are proven in Appendix B. The associated dynamics can be illustrated using a phase diagram, as shown in Figure 4. The illustrated trajectory starts in the lower right portion of the phase plane and cycles toward the interior steady state. These dynamics apply only in the strictly positive quadrant (i.e. where experience and investment opportunities are strictly positive) and are 'local' in the sense that they apply in the neighborhood of the interior steady state.

As can be seen from expression (10), lower values of experience depreciation make cyclical behavior more likely. In other words, if learning-by-doing is important in the sense that experience does not depreciate rapidly, then cycling is more likely. Similarly, inspection of expression (10) combined with conditions (C1) and (C2) also indicates that cyclical patterns of venture capital investment are favored if β is large (i.e. by weaker decreasing returns of investment to experience) or if ρ is large (i.e. by more rapid proportional growth of opportunities).

The effect of δ can be understood by noting that if δ is small then experience accumulates relatively rapidly, leading to high investment and rapid depletion of the pool of opportunities. This depletion then leads to a small pool of remaining unexploited opportunities, which leads to low investment, allowing the pool of opportunities to regenerate and giving rise to a cycle. If δ is large then experience accumulates less rapidly and cycling is less inclined to arise.

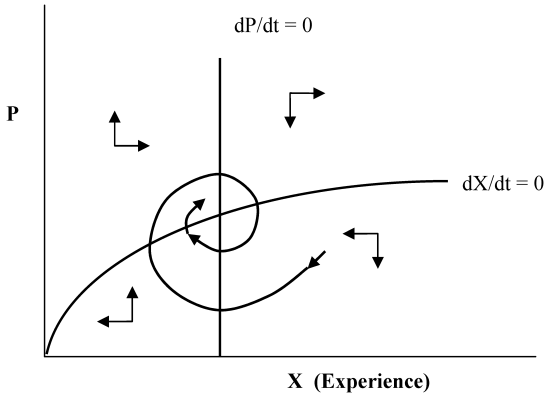

FIGURE 4 Cyclical dynamics.

Parameter β, which must lie between 0 and 1, also has an interesting effect. Higher levels of β imply, other things equal, higher investment and more depletion of P, leading subsequently to low investment and the resulting lower depletion rates as the system cycles toward a steady state. Lower levels of β make this over-shooting less likely. The growth rate of opportunities, ρ, also has a surprising effect. High growth rates do not lead to monotonic or explosive growth of opportunities. Instead they have a strong indirect effect on investment, causing investment to grow rapidly, inducing depletion of opportunities and, ultimately, a convergent cycle. The subtlety of these effects illustrates the interesting and often surprising nature of interactive dynamics.

Our discussion so far applies to local behavior of the system given by Eqs. (5′) and (6′) in the neighborhood of the interior steady state. Global behavior of the linearized system is similar except that we have to be concerned about hitting the horizontal axis ($P = 0$), in which case the system will converge toward the trivial steady state at the origin. We do not view this case as economically meaningful, but we mention it for completeness.

The dynamics of the linearized system will approximate the dynamics of the full nonlinear system very closely in the neighborhood of the steady states (and are precisely correct asymptotically). For points well away from the steady state, the approximation is less precise. However, our simulations of the full system indicate that the linearized system provides a qualitatively correct and close quantitative approximation to the full nonlinear system for a wide range of possible starting points in the phase plane.

5 SIMULATIONS

We now demonstrate that the model described in Sections 3 and 4 is consistent with the three major empirical facts illustrated in Section 2. The first stylized fact documented in Section 2 was the aggregate venture capital cycle of the past 12 years. In Figure 5, we simulate cycles for three industries and then add up the total investment to generate a cycle for total VC investment. The industries are identical except that they start with different levels of associated VC experience. The base case parameters are $\alpha = 0.001$, $\beta = 0.9$, $\rho = 0.05$, and $\delta = 0.05$. The two state variables – experience and opportunities – are indexed to start at 50 and 100,

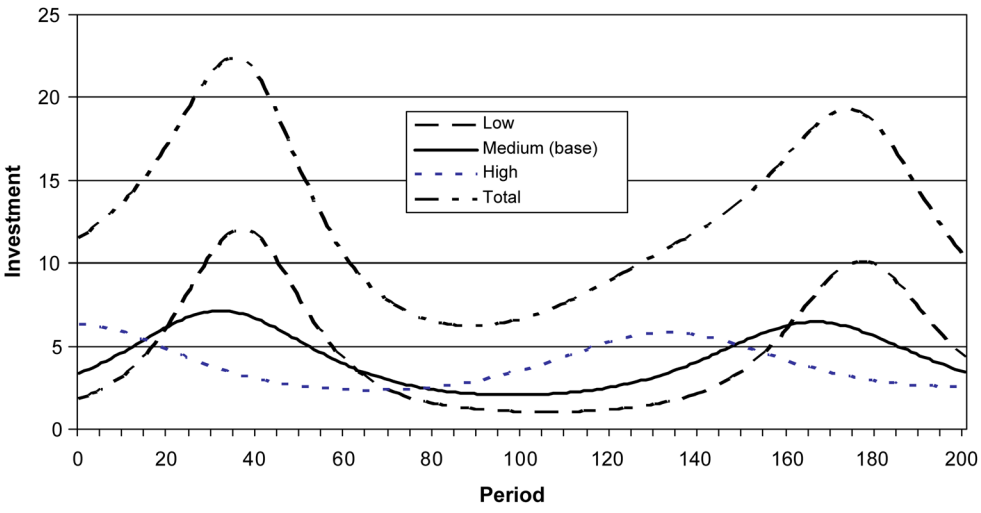

FIGURE 5 Total investment and effects of different initial experience.

respectively. This is the medium initial experience case. The dashed line shows the pattern for an industry with low initial experience of 25. The dotted line shows the 'high' initial experience case, in which initial experience is 100.

Figure 5 exhibits some very interesting properties. First, if we add up the investment levels shown in Figure 5, it is clear that a strong aggregate cycle in venture capital investment emerges. This is consistent with our first stylized fact in Section 2. In addition, Figure 5 shows that simple and modest differences in the amount of initial relevant experience are sufficient to generate large differences in venture capital investment across industries at any given point in time, which is consistent with the second stylized fact illustrated in Section 2.

Figure 5 also illustrates the effects of differential experience. Although high initial experience gives rise to higher initial investment, these high investment levels are relatively short-lived. The high experience sector attracts higher investment levels than the low experience sector only for a short time. This is caused by the fact that the sector with high initial experience attracts high levels of initial investment, but these high investment levels deplete the stock of available opportunities as VCs quickly 'use up' many of the available opportunities.

One measure of the severity of the transition is the 'standardized amplitude' of the cycle, as given by the ratio of amplitude to frequency. In this case, lower initial experience is associated with a steeper transition as measured by the standardized amplitude. Initial experience levels of 25, 50, and 100 lead to standardized amplitudes of about 690, 270, and 250, respectively. All three trajectories gradually approach the same steady state, but the rate of approach is slow enough that the behavior of the system over relevant time horizons is dominated by cyclical (or oscillating) adjustment.

Figure 6 illustrates the share of total investment for each of the three industries shown in Figure 5. The cycling of industry shares is consistent with the third stylized fact of Section 2. Using only variation in the initial experiences of the VCs, and not exploiting any flexibility in other parameter values, it is possible to generate qualitatively similar patterns to the actual venture capital investment. By using the available flexibility with other parameters, it is, of course, possible to replicate the actual patterns of investment even more closely.

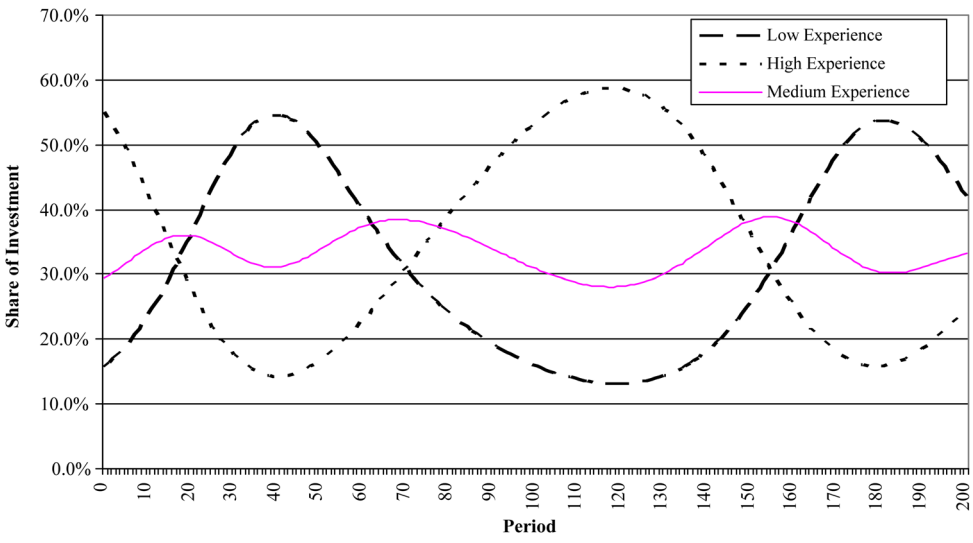

FIGURE 6 Venture capital investment shares.

Our actual data, as shown in Figures 1 and 2, covers only 12 years, so it is difficult to draw inferences about long-run dynamics. Regrettably, reliable data before 1995 is not available. Still, careful inspection of Figure 2 suggests that the relative share of each of the three industries considered (biotechnology, software, and telecommunications) has gone through approximately two full cycles over this 12-year period. Our simulation in Figure 6, covering 200 'periods' covers $\sim 2\frac{1}{2}$; full cycles for each industry. Therefore, our simulation would approximate the data in Figures 1 and 2 if a simulation 'period' were approximately one month.

One can also simulate (not shown) cross-industry differences in the growth rate of opportunities. The standardized amplitude of the cycle is highest for industries with the highest such growth rates. This suggests that industries like biotechnology, software, and telecomunications, which presumably have high opportunity growth rates, would have stronger cycles in both absolute and relative terms than other sectors. This is what we observe. In simulations, the high growth rate industry has persistently high relative investments and the medium growth rate industry is persistently intermediate, demonstrating the persistence effects emphasized in the introduction to this paper. This is also consistent with the second stylized fact in Section 2.

6 DISCUSSION AND CONCLUDING REMARKS

The primary contribution of this paper is to present a model that can explain some of the major characteristics of venture capital investment dynamics. This model can explain both high investment concentration by industry and 'boom and bust' industry-level investment dynamics based primarily on the role of experience or learning by doing by VCs. VCs favor investing in industries where they have significant experience. However, the resulting high levels of investment 'use up' the available unexploited opportunities at a rapid rate, and therefore tend to deplete the pool of unexploited opportunities. This depletion effect tends to reduce subsequent investment, allowing the pool of opportunities to be replenished through natural growth of opportunities due to underlying scientific progress.

The dynamic interaction between experience and the pool of investment opportunities can, for plausible parameter values, give rise to cyclical investment dynamics for a particular industry and for venture capital investment as a whole. However, those industries with a faster natural growth rate of opportunities will tend to have consistently higher venture capital investment rates than other industries. Although venture capital investment dynamics are also influenced by other factors, we suggest that endogenous dynamics of the type demonstrated here are worthy of emphasis in discussions of venture capital investment patterns.

One important insight of our analysis is that venture capital investment dynamics have elements in common with classic predator–prey systems. Investment opportunities in a given industry are like a particular species of prey. These opportunities are subject to natural growth but can be depleted by the actions of a 'predator'. In this case, VCs are the predators who, when they make an investment, take that opportunity out of the pool of available investments and therefore tend to deplete the pool. The predator–prey view of investment could provide at least a partial explanation of the cycles that are often observed in investment behavior.

It would be valuable to distinguish empirically between endogenous dynamics of the predator–prey type and exogenous shocks as an explanation of cycles. This is challenging, but if models with endogenous dynamics were to perform well in a variety of investment contexts, this would increase our confidence in the relevance of the approach.

More sophisticated treatment of some of the simplifications we adopt, although beyond the scope of the present paper, would be valuable. For example, we assume that VCs deplete the pool of available opportunities when they make investments. Alternatively, venture capital investment might induce a positive supply response: research that creates opportunities might

be encouraged by venture capital. This would give rise to a dynamic structure in which venture capital might be a necessary catalyst for development of industry clusters, which in turn generate more funds for further venture capital investment. Such a structure could be captured by replacing Eq. (6) with the equation of motion $\mathrm{d}P/\mathrm{d}t = \rho P + \gamma I$ (with $\rho > 0$ and $\gamma > 0$) over some range. Such positive reinforcement would be unlikely to go on forever, and presumably, predatory effects would dominate eventually (i.e. γ would turn negative), but a rich array of dynamic patterns is possible.

A second extension would be to explicitly consider exogenous uncertainty. We suppress the role of uncertainty by assuming risk neutrality and by interpreting financial flows as expected values. It would be possible to introduce uncertainty explicitly into equations of motion (5) and (6). This might capture random breakthroughs in research that cause the pool of opportunities to increase by more than the normal rate, or it might capture the effect of exogenous random fluctuations in interest rates that affect investment. Formal analysis of such systems is challenging, but they can easily be simulated. They behave much like the deterministic versions. However, random perturbations ensure that the system never settles down to a steady state.

Another extension would be to explicitly consider VC investment portfolio issues. This could be done by incorporating risk aversion by VCs, having an objective function that is concave in profit instead of being equal to profit, and by explicitly introducing sector-specific uncertainty. One could also introduce cross-industry experience effects. In any case, various extensions of our modeling approach can address a number of interesting and empirically relevant considerations.

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APPENDIX A: DATA DEVELOPMENT FOR TABLE I

This appendix describes the details underlying Table I. The venture capital investment data was obtained from PWC Moneytree, which tracks US venture capital investment by industry for 16 industries (and a residual 17th category). Unfortunately, these industries do not match the North

American Industry Classification (NAIC) system used in GDP determination. Therefore, it was necessary to do a concordance between the PWC Moneytree industries and the NAIC system used for GDP. We reduced the set of PWC Moneytree industries from 16 to 14 by combining biotechnology with medical devices and equipment and by combining telecommunications with networking products and services. We then determined the appropriate NAIC codes that should be associated with each PWC industry, closely following the PWC-industry definitions. The resulting concordance is shown in Table A1. As can be seen from the table, the PWC definitions correspond to various levels of aggregation within the NAIC system. We then obtained the relevant GDP data and constructed GDP shares for the PWC industries. GDP data by NAIC classifications was obtained primarily from the US Census as available at www.census.gov

TABLE A1 PWC–NAIC Concordance.

PWC industry	*Sub-industry*	*NAIC code*	
Biotechnology and medical devices	Pharmaceutical and Medicine Manufacturing	3254	
	Synthetics, Agricultural Chemicals	3252, 3253	
	R&D Services in Life Sciences, etc.	54171	
	Medical Equipment and Supplies	3391	
Business products and services	Prof., Sci. & Tech. Serv. (excl. Software Serv.)	54 (5415)	**Q12**
	Management of Enterprises	55	
	Admininistration and Support Services	56	
Computers and peripherals	Computer & Peripheral Man.	3341	**Q12**
	Computer Whol.	4234	**Q12**
Consumer products and services	Accommodation and Food Services	72	
	Other Services (Auto Serv., Personal Serv., etc.)	81	**Q12**
	Food, Clothing, Accessory Manufacturers	31	
	Furniture	337	
Electronics and instrumentation	Navigational, Measuring, Electronic, and Control Instruments Manufacturing	3345	
	Electrical Equipments Manufacturers	335	
Financial services	Finance and Insurance	52	
	Real Estate and Rental and Leasing	53	
Healthcare services	Health Care and Social Assistance	62	
IT services	Internet Publication and Broadcasting	516	
	Internet Providers, Data-Processing, etc.	518	
Industrial and energy	Agriculture, Forestry, Fishing, and Hunting	11	
	Mining	21	
	Utilities	22	
	Construction	23	
	Wood Products, Paper, and Printing	321, 322, 323	
	Petroleum and Coal Production	324	
	Plastics and Rubber	326	
	Nonmetal mineral products (glass, clay, concrete, etc.)	327	
	Metal and Machine Manufacturing	331, 332, 333	
	Transportation Equipment Manufacturing	336	
	Transportation	48	
	Warehousing and Post	49	
Media and entertainment	Arts, Entertainment, and Recreation	71	
	Motion Picture and Sound Recording	512	
	Broadcasting	515	
Retailing and distribution	Wholesale Trade (excl. computers)	42 (4234)	
	Retail Trade	44, 45	
Semiconductors	Semiconductor and other related manufactuers	3344	
Software	Software Publishers	5112	
	Software Services	5415	
Telecommunications and networking	Communications Equipment Manufacturing	3342	
	Telecommunications	517	
	Networking Equipment (fiber optics, etc.)	3346, 3359	

APPENDIX B: TRANSITIONAL DYNAMICS (SECTION 4.2)

This appendix proves the characterization of transitional dynamics set out in Section 4.2. We define u as the vector of deviations from steady state (X^*, P^*).

$$u = (u_X, u_P) = (X - X^*, P - P^*) \tag{A1}$$

Note that $\mathrm{d}u/\mathrm{d}t = (\mathrm{d}X/\mathrm{d}t, \mathrm{d}P/\mathrm{d}t) = (\alpha X^\beta P - \delta X, \rho P - \alpha X^\beta P)$. Using a Taylor series expansion for $\mathrm{d}u/\mathrm{d}t$ around (X^*, P^*), it can be shown (as in Boyce and de Prima (2005, pp. 506–507)) that

$$\left(\frac{\mathrm{d}u}{\mathrm{d}t}\right)' = J(X^*, P^*)u' + R(X, P)', \tag{A2}$$

where $(\mathrm{d}u/\mathrm{d}t)'$ is the transpose of $\mathrm{d}u/\mathrm{d}t$ and is therefore a column vector. J is the Jacobian matrix of first-order partial derivatives of $\mathrm{d}X/\mathrm{d}t$ and $\mathrm{d}P/\mathrm{d}t$ with respect to X and P, and $R(X, P)$ is a remainder of higher order terms that can be ignored near $u = 0$. J is evaluated at (X^*, P^*). Denoting the components of J in the normal way (J_{11}, etc.), we can re-write equation (A2) as

$$\left(\frac{\mathrm{d}u}{\mathrm{d}t}\right)' = \begin{bmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{bmatrix} \begin{bmatrix} u_x \\ u_p \end{bmatrix}. \tag{A3}$$

Since $\mathrm{d}u/\mathrm{d}t = (\mathrm{d}X/\mathrm{d}t, \mathrm{d}P/\mathrm{d}t)$ and $J(X^*, P^*)u'$ is a linear approximation of $(\alpha X^\beta P - \delta X, \rho P - \alpha X^\beta P)$ around the steady state, Eq. (A3) is the linearized version of the system of differential equations defined in Eqs. (5') and (6') in the main text. A system of differential equations as given by Eq. (A3) has a general solution of the form

$$u(t) = c_1 \upsilon_1 \mathrm{e}^{\lambda 1 t} + c_2 \upsilon_2 \mathrm{e}^{\lambda 2 t}, \tag{A4}$$

where c_1 and c_2 are constants, $\lambda 1$ and $\lambda 2$ the eigenvalues of coefficient matrix J, and υ_1 and υ_2 the corresponding eigenvectors. The qualitative properties of the system depend on whether $\lambda 1$ and $\lambda 2$ are real or complex. If $\lambda 1$ and $\lambda 2$ are complex numbers, then cyclical dynamics emerge.

The eigenvalues are the solutions of the matrix equation $J - \mathrm{ID}\upsilon = 0$, where ID is the identity matrix and υ the vector of eigenvalues. Therefore, the eigenvalues can be obtained as values of υ for which the determinant $|J - \mathrm{ID}\upsilon|$ is 0. The equation obtained by setting this determinant to 0 is referred to as the characteristic equation. The elements of J can be obtained by taking partial derivatives of Eqs. (5') and (6') with respect to X and P.

$$J_{11} = \alpha\beta X^{\beta-1} P - \delta, \quad J_{12} = \alpha X^\beta, \quad J_{21} = -\alpha\beta X^{\beta-1} P, \quad J_{22} = \rho - \alpha X^\beta. \tag{A5}$$

For the trivial steady state at $(X, P) = (0, 0)$, the Jacobian matrix becomes

$$J = \begin{bmatrix} -\delta & \alpha \\ 0 & \rho \end{bmatrix}. \tag{A6}$$

For this case, with a zero off-diagonal element, the eigenvalues are simply the diagonal elements: $-\delta$ and ρ. The first of these eigenvalues is a negative real number and the second is a positive real number (using condition (C1)). This implies that (0,0) is an unstable steady state, which means that almost all trajectories starting near (0,0) lead away from it. This steady state can be approached only if $P = 0$, in which case X declines toward 0, which is the lower bound for X.

The more interesting steady state is the interior steady state (X^*, P^*). Substituting Eqs. (7) and (8) into Eq. (A5) yields the following Jacobian matrix.

$$J = \begin{bmatrix} \delta(\beta - 1) & \rho \\ -\beta\rho & 0 \end{bmatrix} \tag{A7}$$

Subtracting IDυ from J and taking the determinant generates characteristic equation:

$$\upsilon^2 + \upsilon\delta(1 - \beta) + \rho\beta\delta = 0. \tag{A8}$$

Depending on the values of parameters α, β, ρ, and δ, the eigenvalues may be real or complex. The main results concerning the local behavior of the linearized system are reported in Proposition 1. The solutions to Eq. (A8) are the standard solution for a quadratic equation:

$$\upsilon = \frac{(\beta - 1)\delta \pm (\delta^2(1 - \beta)^2 - 4\rho\beta\delta)^{1/2}}{2}. \tag{A9}$$

The solutions are complex numbers if the discriminant, $\delta^2(1 - \beta)^2 - 4\rho\beta\delta$, is negative, which occurs if condition (10) holds. The real part of the complex solutions can be written as $((\beta - 1)\delta)/2$, which is negative by condition (C2). Convergent cyclical behavior arises in this case. If Eq. (10) does not hold, then the solutions are real numbers. Since $4\rho\beta\delta > 0$, we must have $(\delta^2(1 - \beta)^2 - 4\rho\beta\delta)^{1/2} < |(\beta - 1)\delta|$. Using this fact in Eq. (A9), and recalling that $\beta < 1$ (by (C2)) shows that both real roots must be negative. This implies monotonic convergence toward the interior steady state. Thus, the system converges to the interior steady state regardless of whether or not Eq. (10) holds.