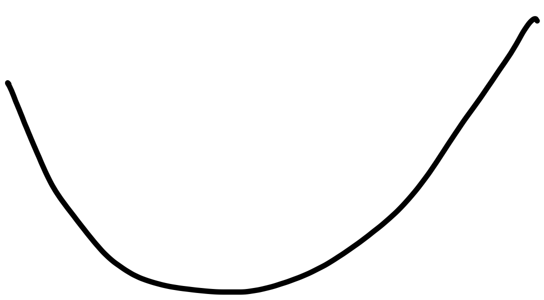
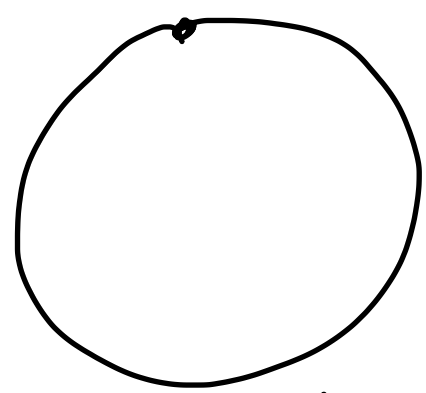
Green's Theorem

- States that the line integral of $\mathbb{F}(x,y)$ around a simple closed curve is the same as the double integral of $\nabla \times \mathbb{F}$ with the boundary:

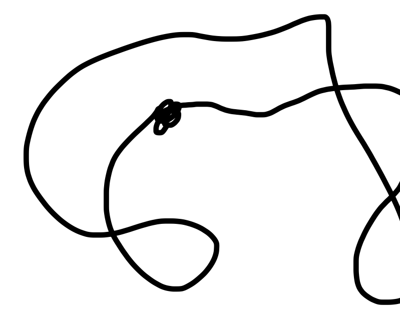
Simple curves.



simple curve



simple closed curve

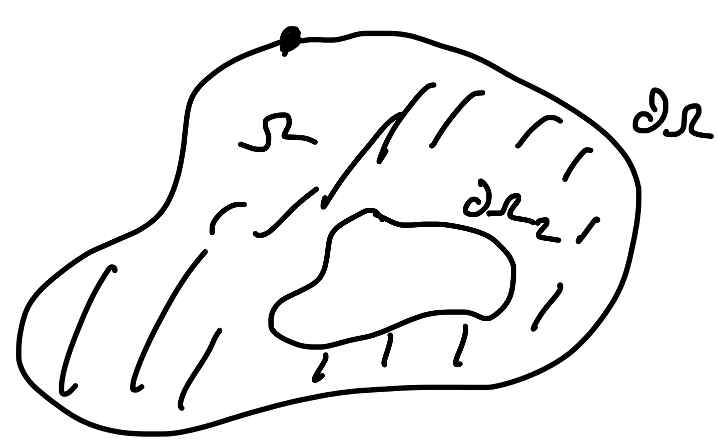


not a simple closed curve.

Definitions.

$\partial\Omega$ ~ boundary.

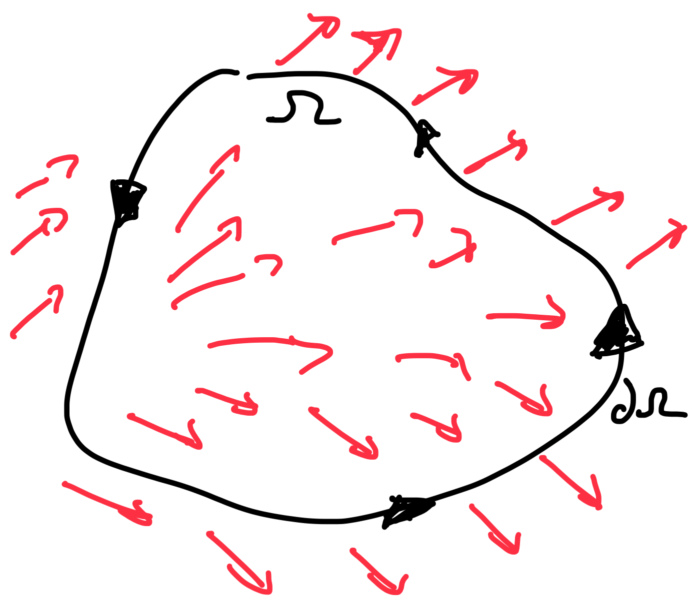
or



where $\partial\Omega = \partial\Omega_1 \cup \partial\Omega_2$
(union of)

Orientation

→ Let's say that I'm interested in some boundary Ω with a vector field $\mathbb{F}(x,y)$.



"Orientation" tells me how we march along the closed curve $\partial\Omega$, i.e. Positive = counterclockwise (c.c.w)
Negative = clockwise. (c.w)

→ Another way to look at this is for positive orientation, $d\mathbf{r}$ moves counterclockwise for $a \le t \le b$.

• With this definition, we can write Green's Theorem :

$$\boxed{\int_{\partial\Omega} \mathbb{F} \cdot d\mathbf{r} = \iint_{\Omega} \nabla \times \mathbb{F} \, dA.}$$

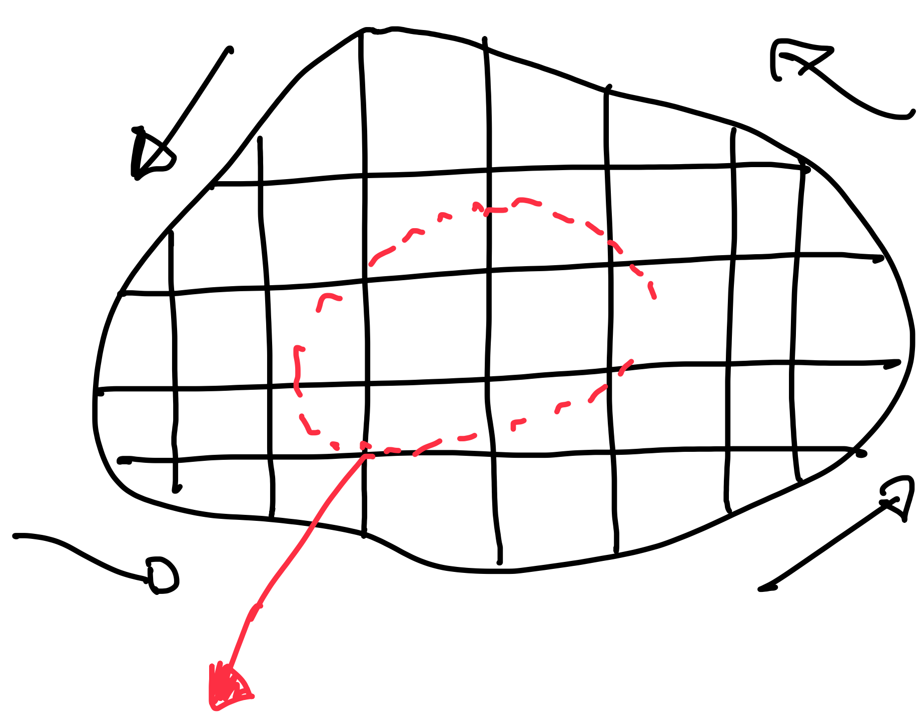
- We can see that firstly, F_x & F_y need to be continuous and differentiable, but we also can see that:

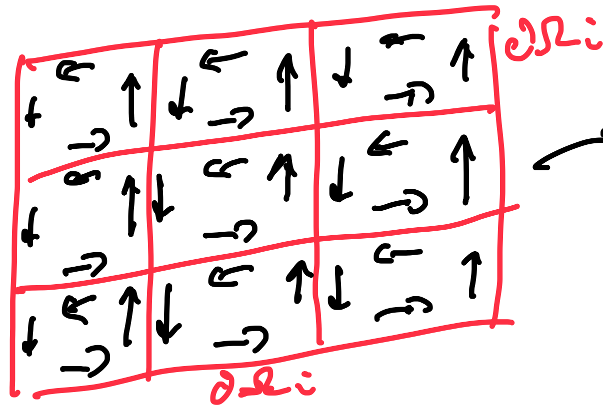
$$\int_{\partial\Omega} \mathbb{F} \cdot d\mathbf{r} > 0$$ if $\mathbb{F}$ (on average) is along the direction of travel dir

$$\int_{\partial\Omega} \mathbb{F} \cdot d\mathbf{r} < 0$$ if $\mathbb{F}$ (on average) is against the direction of travel dir

- We also see that a counterclockwise rotation within Ω and on $\partial\Omega$ is when $\boxed{\nabla \times \mathbb{F} > 0.}$

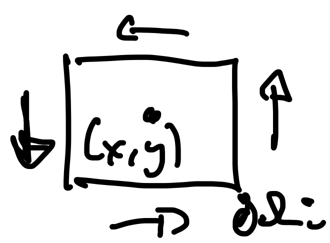
→ Imagine cutting our domain Ω into tiny pieces for a vector field that generally rotates c.c.w.





The vectors cancel eachother on each horizontal / vertical cell interphase

- Now, for each cell we just need the line integral to arrive at Green's Theorem:

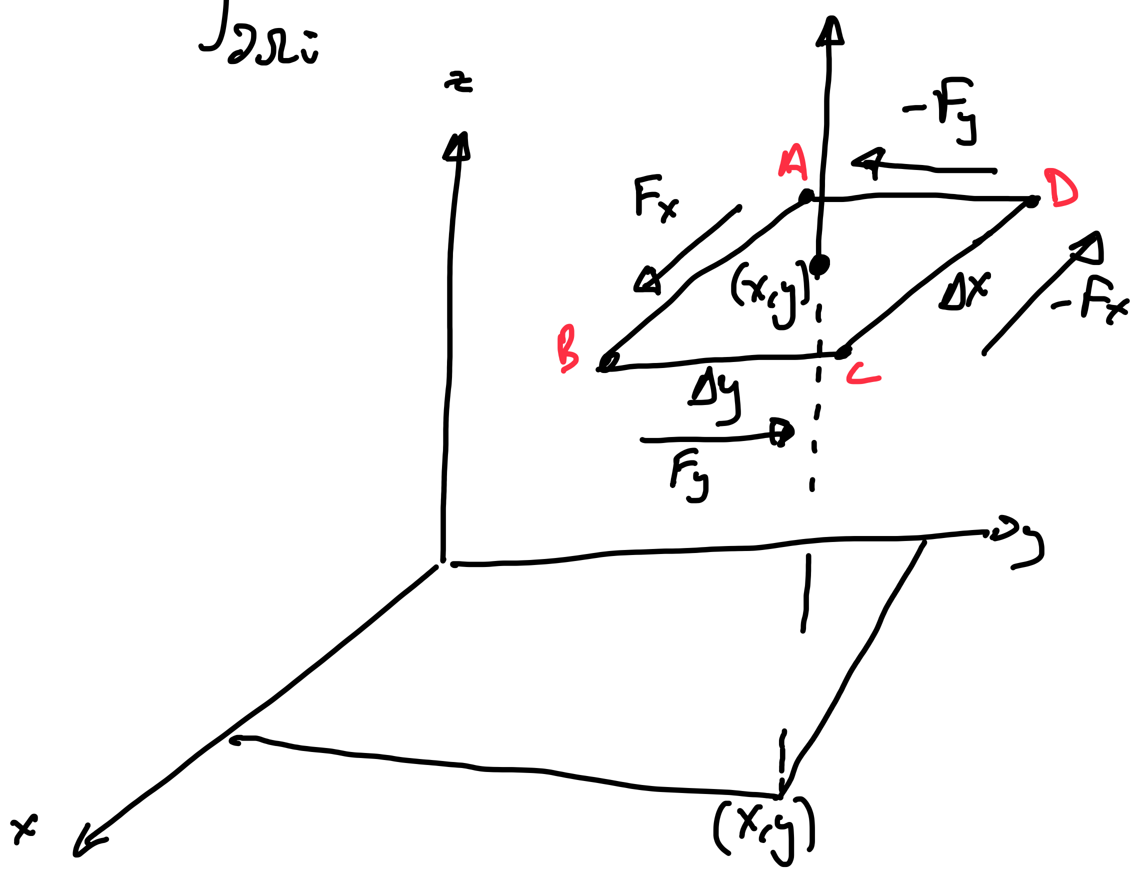


$$\int_{\partial\Omega_i} \mathbb{F}\cdot d\mathbf{r} \approx \nabla\times\mathbb{F}\,(\text{Area}).$$

And, summing over all cells, we get:

$$\boxed{\sum_{i=1}^{N}\int_{\partial\Omega_i} \mathbb{F}\cdot d\mathbf{r} = \int_{\partial\Omega} \mathbb{F}\cdot d\mathbf{r} = \iint_{\Omega} \nabla\times\mathbb{F}\, dA.}$$

→ Why is $\int_{\partial\Omega_i} \mathbb{F}\cdot d\mathbf{r} = \nabla\times\mathbb{F}\,(\text{Area})$?



Lets do the line integral for $A \to B$.

$\int_A^B \mathbb{F}\cdot d\mathbf{r}$ ⇒ We can use a Taylor series to approximate $\mathbb{F}(x, y-\Delta y/2)$, i.e.

$$F_x(x, y-\Delta y/2) \sim F_x(x,y) - \frac{\Delta y}{2}\frac{\partial F_x}{\partial y} + \frac{1}{2!}\left(\frac{\Delta y}{2}\right)^2 \frac{\partial^2 F_x}{\partial y^2}$$

→ Cutting our approximation off at first order, we're left with:

$$\int_A^B \mathbb{F}\cdot d\mathbf{r} = \left[F_x - \frac{\Delta y}{2}\frac{\partial F_x}{\partial y}\right]\Delta x$$

→ Doing the same thing for C→D, we get:

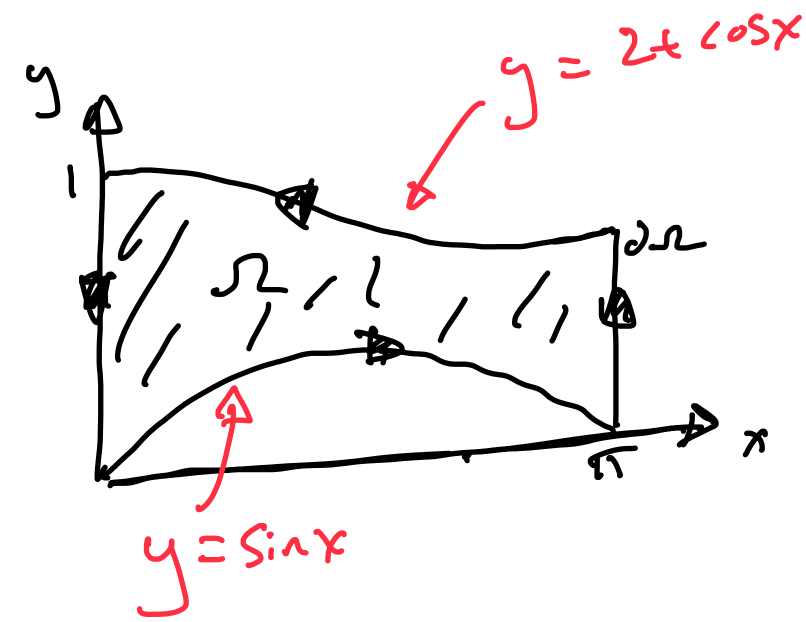
$$\int_C^D \mathbb{F}\cdot d\mathbf{r} = -\left[F_x + \frac{\Delta y}{2}\frac{\partial F_x}{\partial y}\right]\Delta x \quad , \quad \text{and hence}$$

$$\int_{\partial\Omega} \mathbb{F}\cdot d\mathbf{r} = \int_A^B \mathbb{F}\cdot d\mathbf{r} + \int_B^C \mathbb{F}\cdot d\mathbf{r} + \int_C^D \mathbb{F}\cdot d\mathbf{r} + \int_D^A \mathbb{F}\cdot d\mathbf{r}$$

$$\approx \left[\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right]\Delta x \Delta y.$$

$$= \left[\nabla \times \mathbb{F}\right]\Delta x \Delta y.$$

Ex.



$\mathbb{F}(x,y) = \langle e^x, 2x \rangle.$

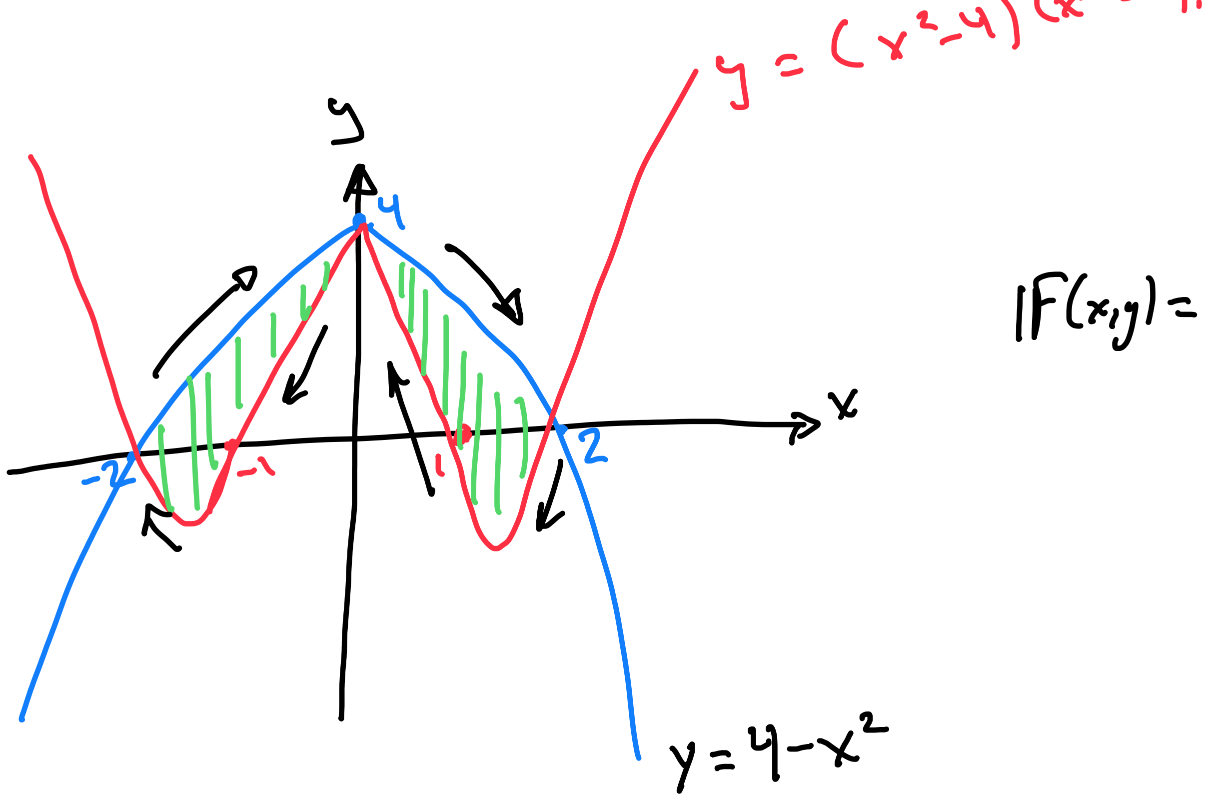
→ Find the circulation of $\mathbb{F}(x,y)$, ie. line integral along $\partial\Omega$.

$$\int_{\partial\Omega} \mathbb{F}\cdot d\mathbf{r} = \iint_{\Omega} \nabla\times\mathbb{F}\, dA.$$

$$= \iint_{\Omega} \frac{\partial}{\partial x}(2x) - \frac{\partial}{\partial y}(e^x)\, dA$$

$$= 2\int_0^{\pi}\int_{\sin x}^{2+\cos x} dA. \quad = 2(\pi - 2)$$

Ex.



$$\mathbb{F}(x,y) = \langle x^2 y, y^2 \rangle.$$

→ Find the circulation of the region that is bounded between the two curves.

For arguments sake, I've oriented the curve clockwise ... The only thing we need to do in the end, is multiply our solution by -1, accounting for the c.w. rotation along the curve.

$$\int_{\partial\Omega} \mathbb{F} \cdot d\mathbf{r} = \iint_{\Omega} \nabla \times \mathbb{F} \, dA$$

$$= \iint_{\Omega} \left[\frac{\partial y^2}{\partial x} - \frac{\partial}{\partial y}(x^2 y) \right] dA$$

$$= \iint_{\Omega} -x^2 \, dA$$

(Now, flip the sign to account for c.w. orientation).

$$\int_{\partial\Omega} \mathbb{F} \cdot d\mathbf{r} = \int_{-2}^{2} \int_{(x^2-4)(x^2-1)}^{4-x^2} x^2 \, dy \, dx = 256 \left[\frac{1}{5} - \frac{1}{7} \right]$$