Review of div & curl.

- Up to now we've seen for a vector function

$\mathbb{F} = \langle F_1, F_2, F_3 \rangle = \delta_1 F_1 + \delta_2 F_2 + \delta_3 F_3$, that:

$$\nabla \times \mathbb{F} = \text{curl } \mathbb{F} = \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z} \right) \delta_1 - \left(\frac{\partial F_3}{\partial x} - \frac{\partial F_1}{\partial z} \right) \delta_2 + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) \delta_3.$$

- We also know that if $\mathbb{F}$ is conservative, then there exists some scalar function such that: $\mathbb{F} = \nabla \phi$, as long as $\mathbb{F}$ is continuous and differentiable.

Q
Are conservative vector fields irrotational?

A
We know they have to be, but let's prove it.

$$\nabla \times \nabla \phi = \begin{bmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial \phi}{\partial x} & \frac{\partial \phi}{\partial y} & \frac{\partial \phi}{\partial z} \end{bmatrix}$$

$$= \left[\frac{\partial^2 \phi}{\partial y \partial z} - \frac{\partial^2 \phi}{\partial z \partial y} \right] \partial_1 - \left[\frac{\partial^2 \phi}{\partial x \partial z} - \frac{\partial^2 \phi}{\partial x \partial z} \right] + \dots$$

$$= 0.$$

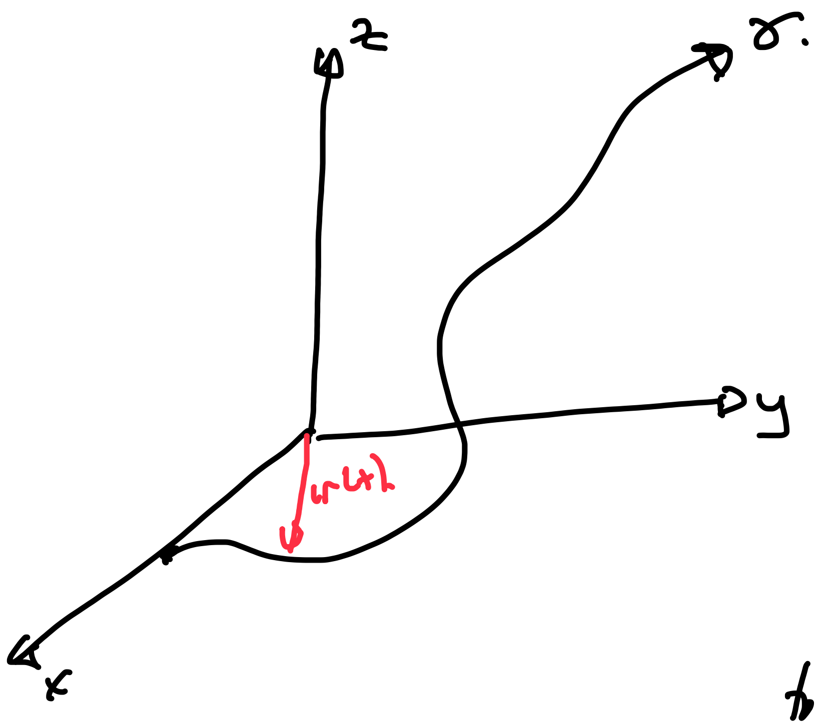
- The curl of a vector field describes the rotation of, for example, a fluid element at any point within a vector / force field. It gives you the magnitude of rotation and the axis of rotation.

<u>Eg.</u> $\mathbf{v} = \langle -y, x, 0 \rangle.$

$\nabla \times \mathbf{v} = \langle 0, 0, 2 \rangle.$

Hence, the rotation is around the positive z-axis with a constant magnitude <u>2</u>, ie rotating counter-clockwise!!

Gradients.



$\mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle.$

Think of gradients as the "total" rate of change.

Ex
If I'm travelling in a temperature field, then the total rate of change of temperature that I feel is:

$$\frac{\partial}{\partial t} T(x,y,z) = \frac{\partial T}{\partial x}\frac{\partial x(t)}{dt} + \frac{dT}{\partial y}\frac{\partial y(t)}{dt} + \frac{\partial T}{\partial z}\frac{\partial z(t)}{\partial t}$$

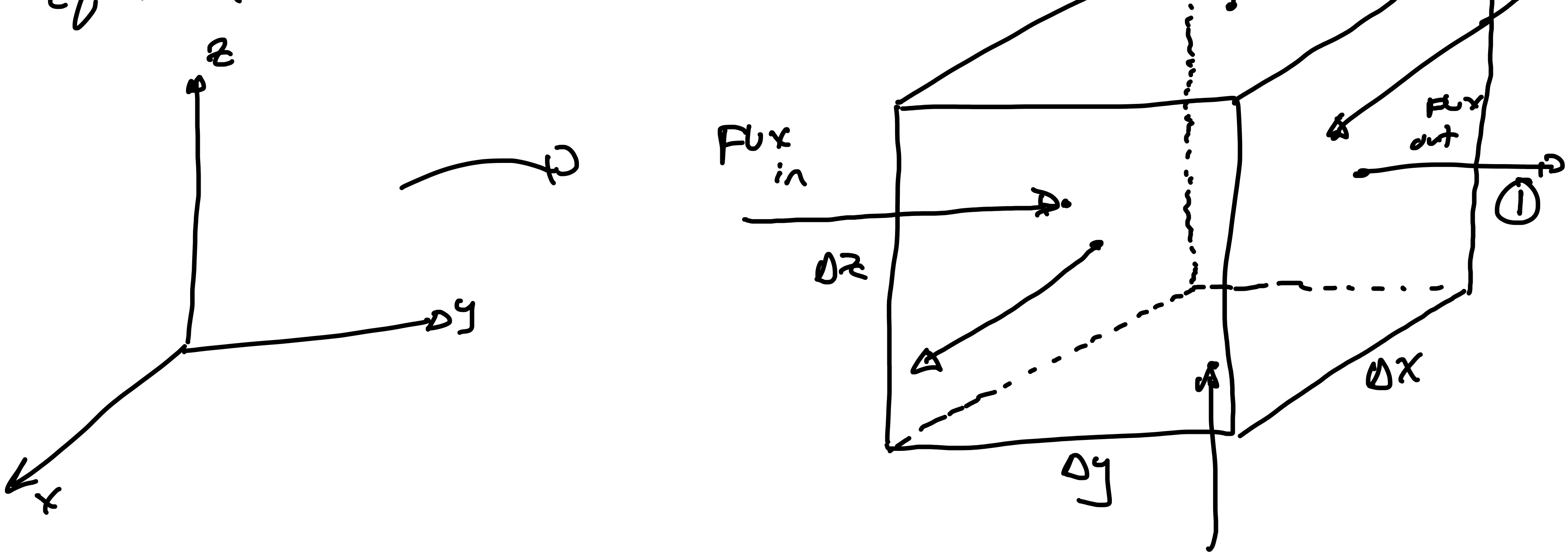
$$= \nabla T \cdot \frac{\partial \mathbf{r}}{\partial t}$$

Divergence

Think of divergence of a vector field as the rate of which fluid is exiting a volume.

ie. $\nabla \cdot \mathbf{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}$ = strength of source.

→ Let's derive this in the context of fluid flowing through a cube, i.e. "the continuity equation"!



If we let, $\mathbb{v} = \langle v_x, v_y, v_z \rangle$, then we can approximate the flux in/out of each face using Taylor's approximation.

If we define the density of fluid as 'ρ', then for face ① we get:

$$\rho v_y(x, y + \Delta y/2, z) = \rho v_y(x, y, z) + \left(\frac{\partial \rho v_y}{\partial y} \frac{\Delta y}{2} \right) + \ldots$$

Carrying this forward for each face and sum all flux in/out, we get.

$$\sum_{M_{in}} \approx \left(\rho v_x - \frac{\partial}{\partial x}(\rho v_x)\frac{\Delta x}{2}\right)\Delta y \Delta z + \left(\rho v_y - \frac{\partial}{\partial y}(\rho v_y)\frac{\Delta y}{2}\right)\Delta x \Delta z$$
$$+ \left(\rho v_z - \frac{\partial}{\partial z}(\rho v_z)\frac{\Delta z}{2}\right)\Delta x \Delta y$$

$$\sum_{M_{out}} \approx \left(\rho v_x + \frac{\partial}{\partial x}(\rho v_x)\frac{\Delta x}{2}\right)\Delta y \Delta z + \left(\rho v_y + \frac{\partial}{\partial y}(\rho v_y)\frac{\Delta y}{2}\right)\Delta x \Delta z$$
$$+ \left(\rho v_z + \frac{\partial}{\partial z}(\rho v_z)\frac{\Delta z}{2}\right)\Delta x \Delta y.$$

We use Reynold's Transport theorem, to obtain.

$$\frac{d}{dt}\int_{\Omega(t)} \rho \, dV = \int_{\Omega(t)} \frac{\partial \rho}{\partial t} \, dV + \underbrace{\int_{\partial\Omega(t)} (\mathbf{v}\cdot\mathbf{n})\rho \, dA}_{= \sum_{M_{out}} - \sum_{M_{in}}} = 0$$

$\Omega(t)$ - volume element.

mass cannot be created or destroyed

$\therefore$ we get!

$$\frac{\partial \rho}{\partial t}\overbrace{\Delta x \Delta y \Delta z}^{\text{volume} = dV} + \frac{\partial(\rho v_x)}{\partial x}\Delta x \Delta y \Delta z + \frac{\partial(\rho v_y)}{\partial y}\Delta x \Delta y \Delta z + \frac{\partial(\rho v_z)}{\partial z}\Delta x \Delta y \Delta z = 0$$

or

$$\boxed{\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho \mathbf{v}) = 0}$$