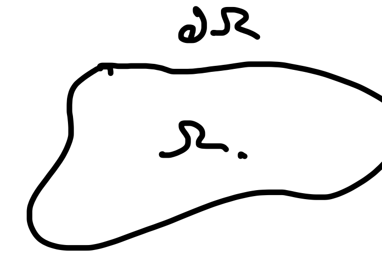
Divergence Theorem.

→ The 2D divergence theorem is to divergence what Green's theorem is to curl.

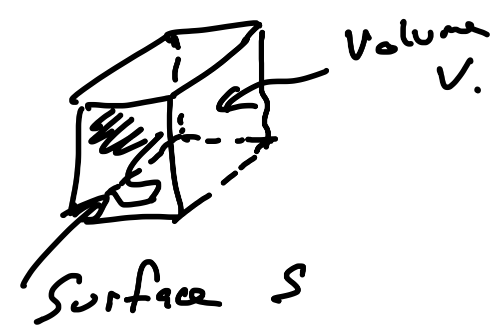
• States that the flux $\mathbb{F}$ through a boundary curve $\partial\Omega$ is the same as the double integral of the $\nabla\cdot\mathbb{F}$ over all Ω.

2D → $\displaystyle\int_{\partial\Omega} \mathbb{F}\cdot\mathbf{n}\, ds = \iint_{\Omega} \nabla\cdot\mathbb{F}\, dA$



3D → $\displaystyle\iint_{S} \mathbb{F}\cdot\mathbf{n}\, d\Sigma = \iiint_{V} \nabla\cdot\mathbb{F}\, dV$

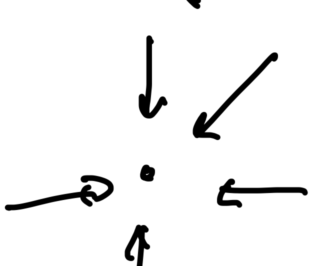
small chunk of the surface "S".



Let's focus on 2D for now...

Ex.

$\nabla\cdot\mathbb{F} < 0$



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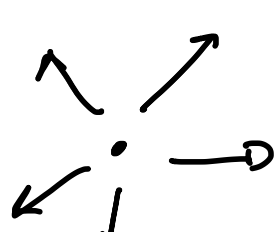
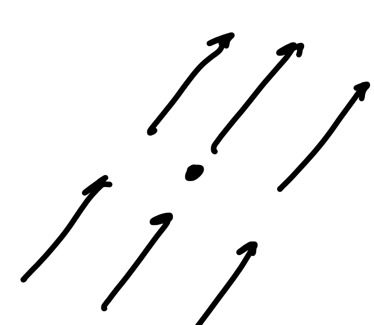


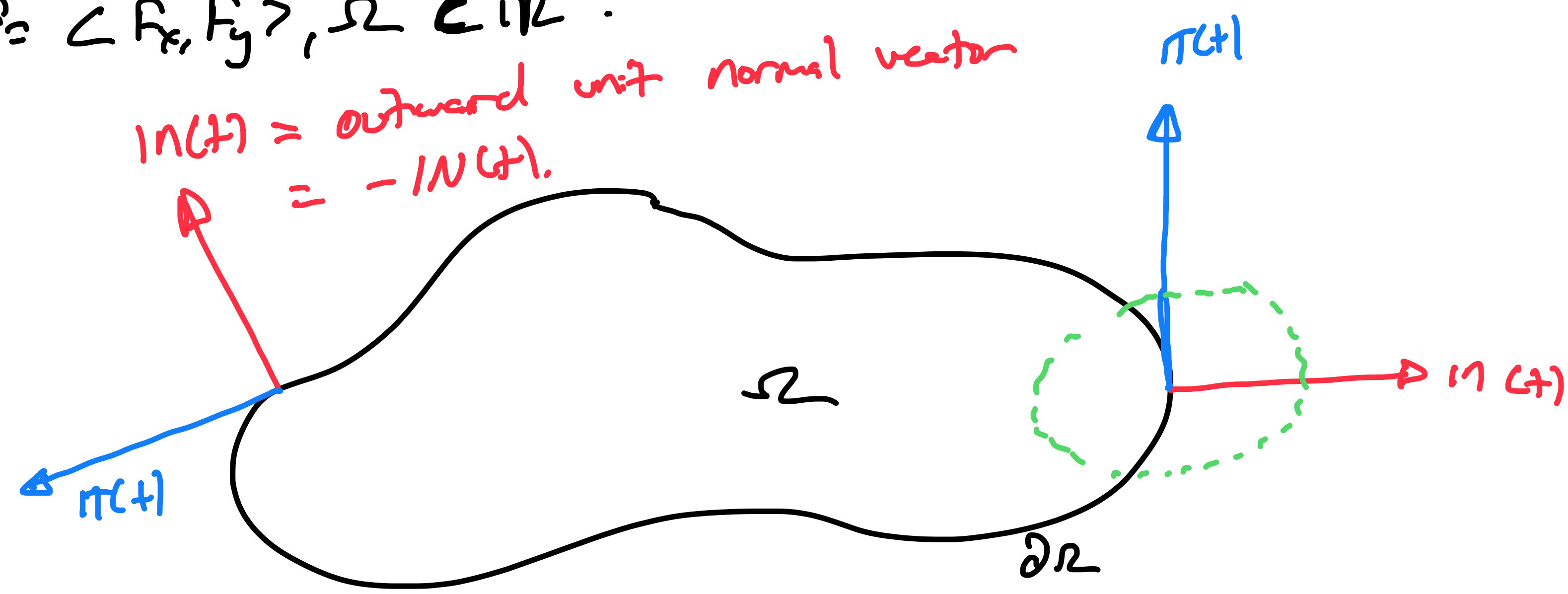
$\nabla\cdot\mathbb{F} = 0$.



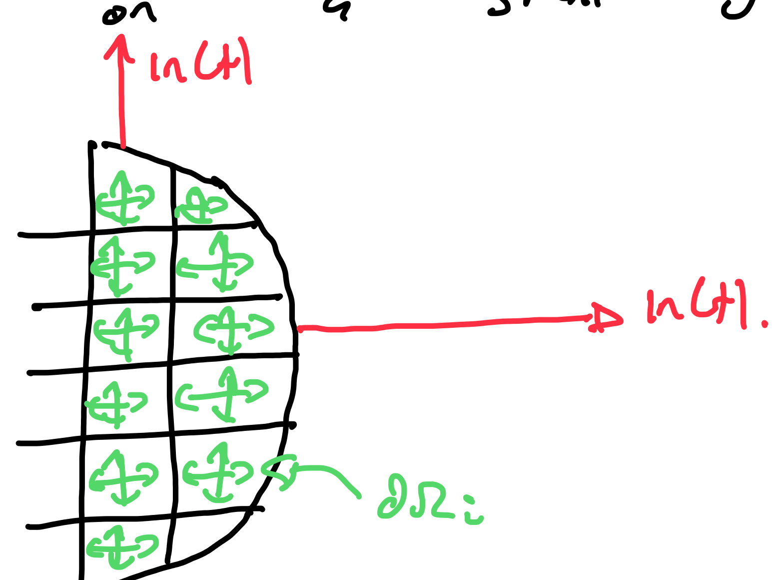
- Derivation is very similar to Green's theorem ...
  Recall breaking up the surface into small pieces?
  Let's try the same approach and see for ourselves how Green's & divergence compare.

- Let's say Ω lies within a vector field $\mathbb{F} = \langle F_x, F_y \rangle$, $\Omega \in \mathbb{R}^2$.

$\mathbb{n}(t)$ = outward unit normal vector
$= -\mathbb{N}(t)$.



- Let's focus on a small region (green).



We can see that again, as in Green's Theorem, the vectors cancel eachother.

Here,

$$\sum_{\partial\Omega_i} \mathbb{F}\cdot\mathbb{n}\, ds = \sum_{\partial\Omega} \mathbb{F}\cdot\mathbb{n}\, ds, \quad \text{and}$$

as $\partial\Omega_i$ approaches zero,

$$\int_{\partial\Omega_i} \mathbb{F}\cdot\mathbb{n}\, ds \approx \nabla\cdot\mathbb{F}\,(\text{Area})$$

$$\therefore \boxed{\int_{\partial\Omega} \mathbb{F}\cdot\mathbb{n}\, ds = \iint_{\Omega} \nabla\cdot\mathbb{F}\, dA}$$

<u>Ex,</u>

Show that $\displaystyle\int_{\partial\Omega} \mathbb{F}\cdot\mathbb{n}\, ds = \iint_{\Omega} \nabla\cdot\mathbb{F}\, dA$

for $\mathbb{F}(x,y) = \langle x, y\rangle$ <u>AND</u> $\mathbb{r}(t) = \langle 2\cos t, 2\sin t\rangle$.

Let's first find:

$\iint_{\Omega} \nabla \cdot \mathbb{F} \, dA$ → Since our surface is a circle,

$dA = \partial x \, \partial y$

$= r \, \partial\theta \, \partial r.$ (Jacobian matrix).

$\therefore \iint_{\Omega} \nabla \cdot \mathbb{F} \, dA = \int_0^{2\pi}\int_0^{2} 2r \, \partial\theta \, \partial r = 8\pi$

<u>Method #2.</u>

$\int_{\partial\Omega} \mathbb{F} \cdot \mathbb{n} \, ds = \int_{\partial\Omega} \mathbb{F} \cdot \mathbb{n} \, |\mathbb{r}'(t)| \, dt.$

$\mathbb{r}'(t) = \langle -2\sin t, 2\cos t \rangle$ → $|\mathbb{r}'(t)| = 2.$

$\mathbb{T}(t) = \langle -\sin t, \cos t \rangle.$

$\mathbb{N}(t) = -\langle \cos t, \sin t \rangle.$

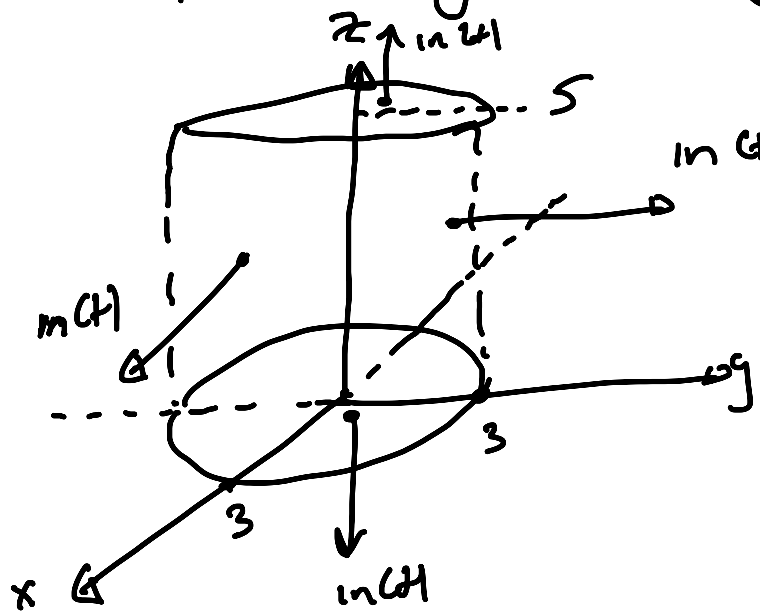
$\mathbb{n}(t) = -\mathbb{N}(t) = \langle \cos t, \sin t \rangle.$

Hence, $\mathbb{F}(\mathbb{r}(t)) \cdot \mathbb{n}(t) = 2(\cos^2 t + \sin^2 t) = 2$

$\therefore 2\int_0^{2\pi} 1 \, (2) \, \partial t = \boxed{8\pi}$

3D example

→ Flux through a cylinder of radius 3 and height 5.



→ Let $\mathbb{F}(x,y) = \langle x^3, y^3, x^3+y^3 \rangle$.

What is the flux of fluid through the cylinder?

$$\iint_S \mathbb{F}\cdot\mathbb{n}\, d\Sigma = \underbrace{\iiint_V \nabla\cdot\mathbb{F}\, dV}_{\text{Easier to calculate.}}$$

$\nabla\cdot\mathbb{F} = 3x^2+3y^2 = 3(r^2)$ ie. $r^2 = x^2+y^2$

$\iiint_V 3r^2\, dV$ → $dV = dx\,dy\,dz$
$= r\,dr\,d\theta\,dz$ ← ie. Jacobian determinant

$$J_f = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = r$$

Hence $3\int_0^5\int_0^3\int_0^{2\pi} r^3\, d\theta\, dr\, dz = \boxed{\frac{1215\pi}{2}}$