Parametrized Surfaces.

- Up to now we've focused mainly on parametrizing lines in terms of one variable, e.g. arclength or some linear variable t (angle, time, etc.).

- Now, we're going to extend this to surfaces, similar to what we did last class with the cylinder.

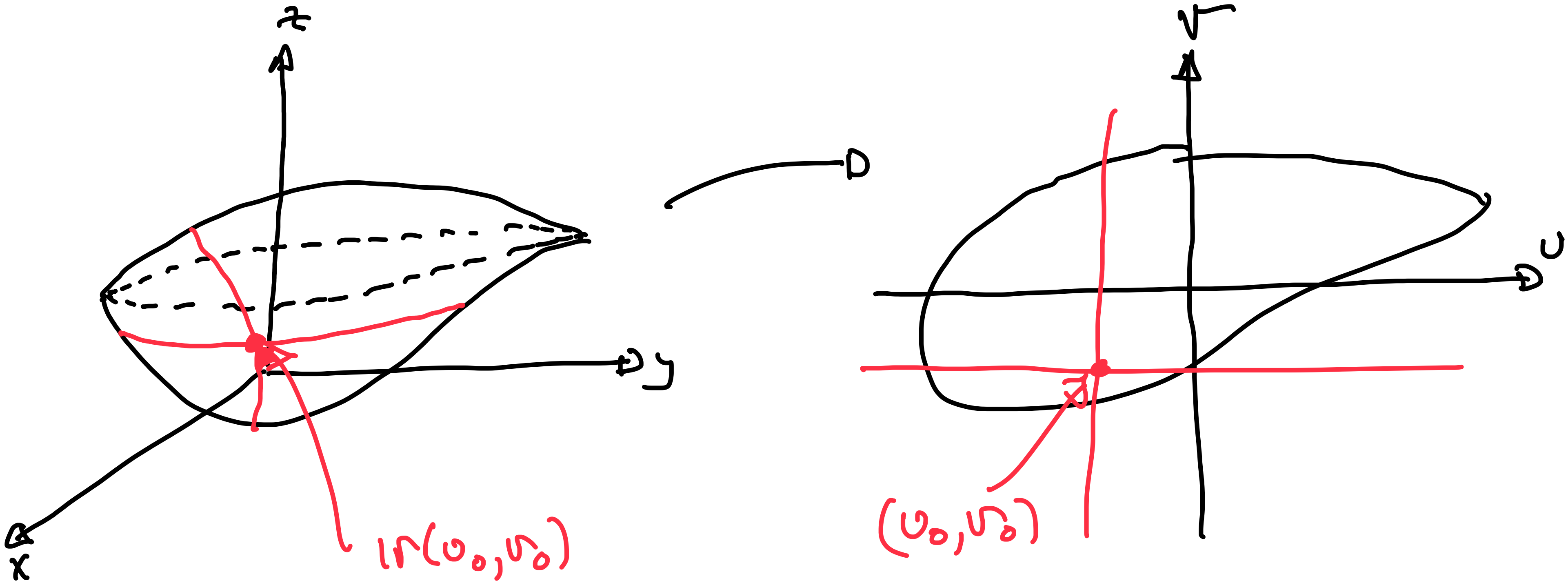
- There are a few ways to do this!

1) Build a function for the surface, for example, a sphere (centered at the origin)!

$$f_s(x, y, z) = x^2 + y^2 + z^2 - R,$$

where we trace the surface of the sphere that's radius R. This is a root finding method to find x, y, z at the surface.

2) Parametrize the surface such that each point is described by two parameters, which we call 'u', and 'v'.

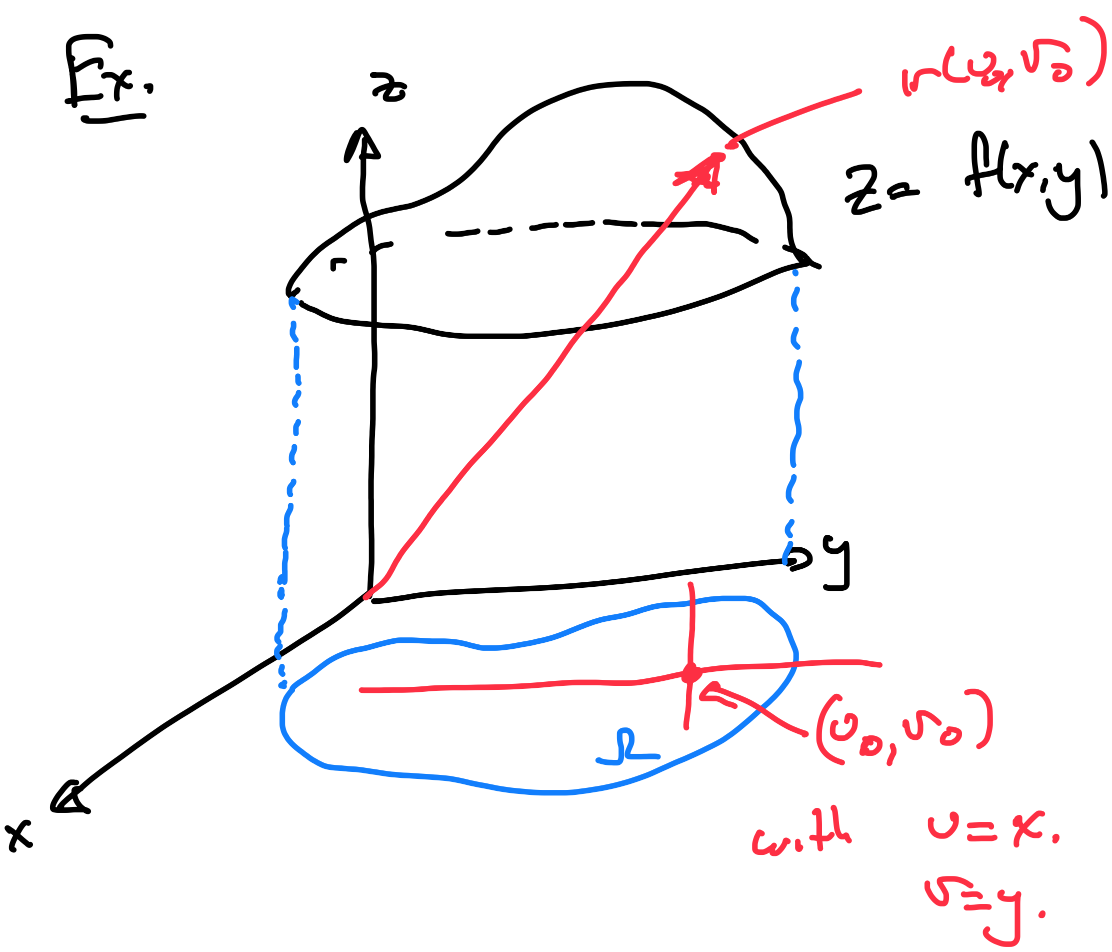


Basically, we're defining the surface such that:

$$\mathbf{r}(u,v) = \langle x(u,v), y(u,v), z(u,v) \rangle \in \mathbb{R}^3$$

$$(u,v) \in \Omega \subset \mathbb{R}^2$$

Ex.



Lets use the example of an arbitrary plane,

ie. $Ax + By + Cz + D = 0$.

$\therefore$ we can rearrange to

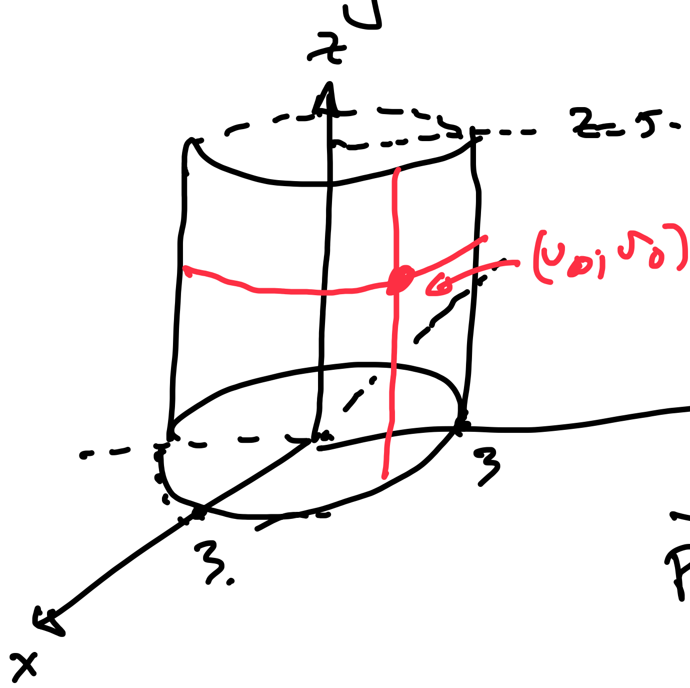
$$z = -\frac{A}{C}x - \frac{B}{C}y - \frac{D}{C} = f(x,y)$$

$\therefore$ A parametrized surface is:

$$\boxed{\mathbf{r}(u,v) = \langle u, v, -\frac{A}{C}u - \frac{B}{C}v - \frac{D}{C} \rangle.}$$

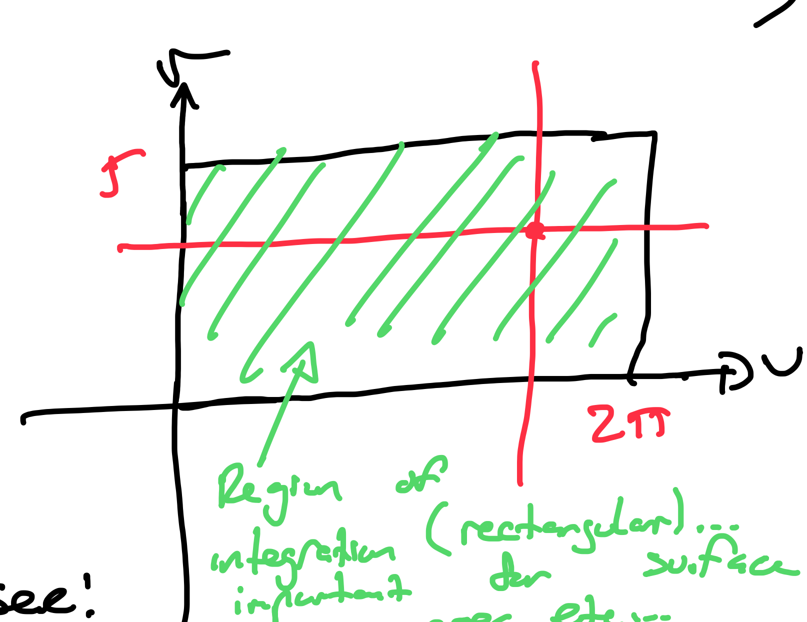
Ex.

A cylinder (from yesterday's class).



$$\left\{ (x,y,z) \in \mathbb{R}^3 : x^2+y^2=3,\ 0 \le z \le 5 \right\}$$

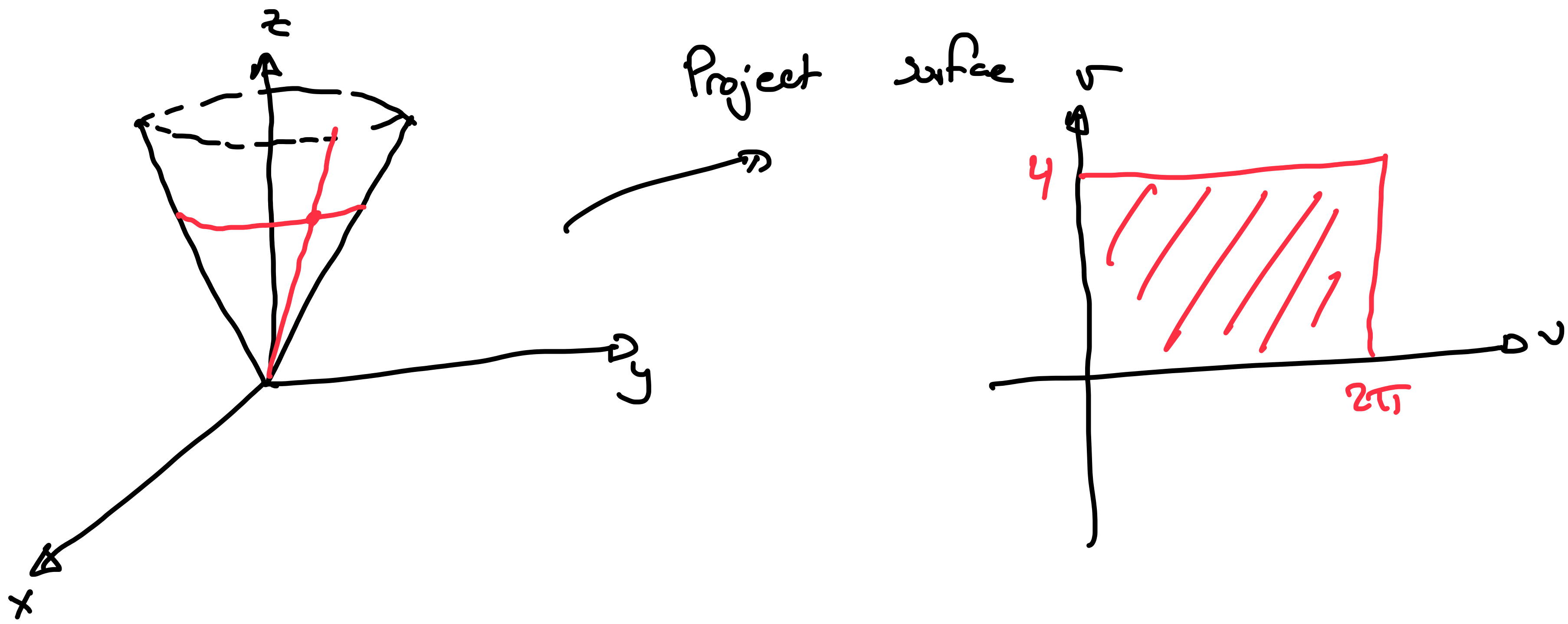
Project onto $\mathbb{R}^2$



With a change of variables, we see!

$$\boxed{\mathbf{r}(u,v) = \langle 3\cos u, 3\sin u, v \rangle,\quad (u,v) \in [0,2\pi] \times [0,5]}$$

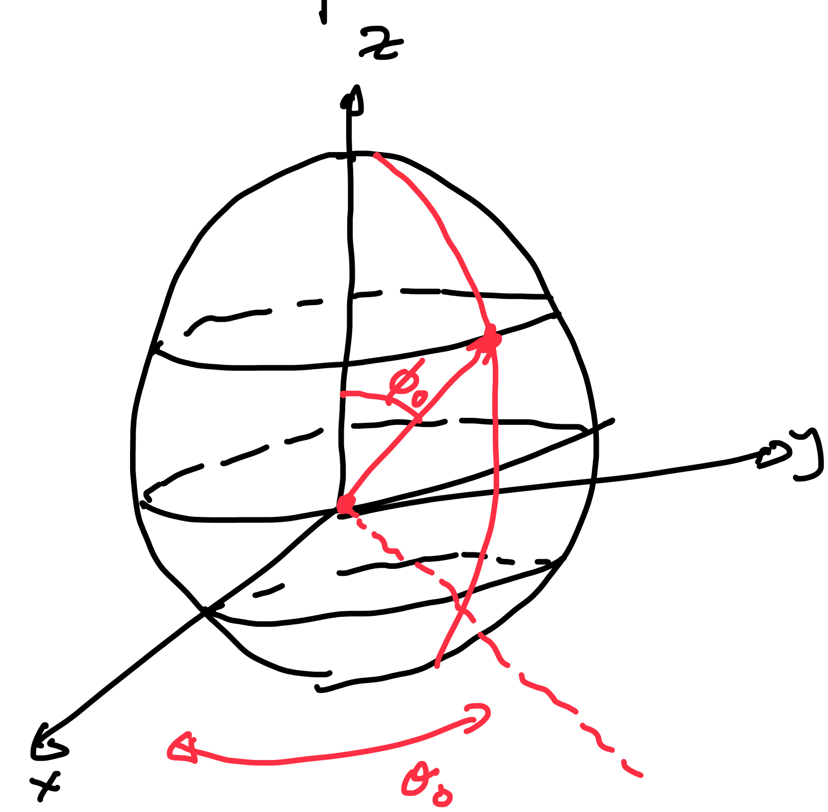
Ex. Cone. $\Rightarrow \{x^2+y^2=z^2, \quad 0\le z\le 4\}$



Here, our parametrized surface is:

$$\boxed{\begin{aligned}\mathbf{r}(u,v) &= \langle v\cos u,\ v\sin u,\ v\rangle. \\ &(u,v)\in[0,2\pi]\times[0,4]\end{aligned}}$$

Ex. Sphere. $\Rightarrow x^2+y^2+z^2=9.$



Option #1 → Spherical coordinates

$\mathbf{r}(\theta, \phi) = \langle 3\cos\theta\sin\phi, 3\sin\theta\sin\phi, 3\cos\phi \rangle.$

let, $u = \theta$, $v = \phi$. , $(u,v) \in [0, 2\pi] \times [0, \pi]$

Option #2 → polar coordinates

Rearrange $x^2+y^2+z^2=9$ to yield.

$x^2+y^2 = 9-z^2$, hence.

$x = \sqrt{9-z^2}\cos\theta$

$y = \sqrt{9-z^2}\sin\theta$

Yielding ⇒ $\boxed{\mathbf{r}(u,v) = \langle \sqrt{9-v^2}\cos u, \sqrt{9-v^2}\sin u, v \rangle.}$

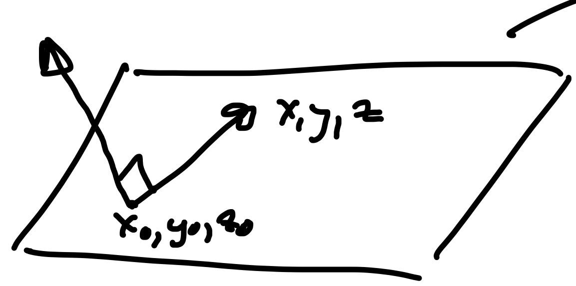
where $(u,v) \in [0, 2\pi] \times [-3, 3]$

Tangent planes.

If you run into complicated surfaces & want to get an idea of what the surface looks like at specific points, you'll need to approximate the surface with a tangent plane

Equation of the plane

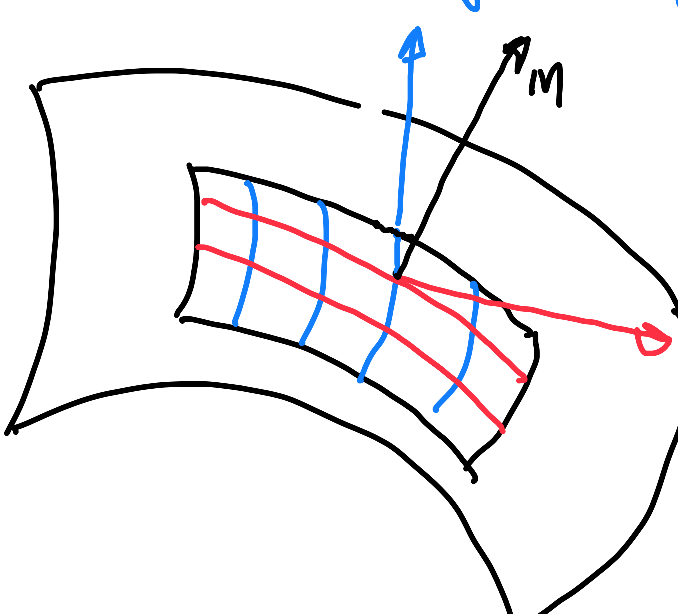
Recall that:



$$\boxed{n \cdot \langle x-x_0, y-y_0, z-z_0 \rangle = 0}$$

Ex.

T_v - tangent vector with respect to v



T_u - tangent vector with respect to u.

Recall that for a parametrized surface:

$\mathbf{r}(u,v) = \langle x(u,v), y(u,v), z(u,v) \rangle$, then

for any point on the surface, e.g. $(x_0, y_0, z_0) = \mathbf{r}(u_0, v_0)$,

the tangent vectors to the surface are:

$$\mathbf{T}_u = \frac{\partial}{\partial u} \mathbf{r}(u, v_0) \Big|_{u=u_0} = \left\langle \frac{\partial x}{\partial u}(u_0,v_0), \frac{\partial y}{\partial u}(u_0,v_0), \frac{\partial z}{\partial u}(u_0,v_0) \right\rangle.$$

&

$$\mathbf{T}_v = \frac{\partial}{\partial v} \mathbf{r}(u_0, v) \Big|_{v=v_0} = \left\langle \frac{\partial x}{\partial v}(u_0,v_0), \frac{\partial y}{\partial v}(u_0,v_0), \frac{\partial z}{\partial v}(u_0,v_0) \right\rangle.$$

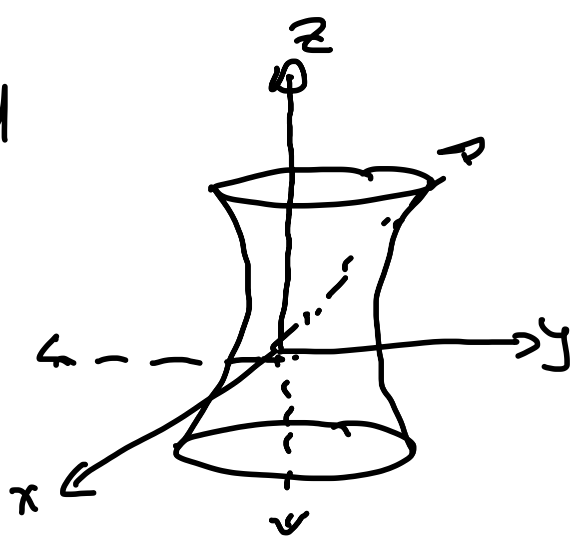
Hence, our normal vector is just:

$$\boxed{\mathbf{n} = \mathbf{T}_u \times \mathbf{T}_v}$$

Ex

Find the tangent plane to the surface:

$x^2 + y^2 = z^2 + 1$, ie a hyperboloid



Best choice to parametrize is with a polar system:

$$\mathbb{r}(u,v) = \langle f(v)\cos u,\ f(v)\sin u,\ v \rangle.$$

$\therefore$ for the hyperboloid, we get:

$$\mathbb{r}(u,v) = \langle \sqrt{v^2+1}\ \cos u,\ \sqrt{v^2+1}\ \sin u,\ v \rangle.$$

Let's now find $\mathbb{T}_u$ & $\mathbb{T}_v$.

$$\mathbb{T}_u = \frac{\partial}{\partial r}\mathbb{r}(u,v) = \mathbb{r}_u = \langle -\sqrt{v^2+1}\ \sin u,\ \sqrt{v^2+1}\ \cos u,\ 0 \rangle.$$

$$\mathbb{T}_v = \frac{\partial}{\partial r}\mathbb{r}(u,v) = \mathbb{r}_v = \langle v(1+v^2)^{-\frac{1}{2}}\cos u,\ v(1+v^2)^{-\frac{1}{2}}\cos u,\ 1 \rangle.$$

Here, $\mathbb{n}(u_0,v_0) = \mathbb{T}_u(u_0,v_0) \times \mathbb{T}_v(u_0,v_0)$

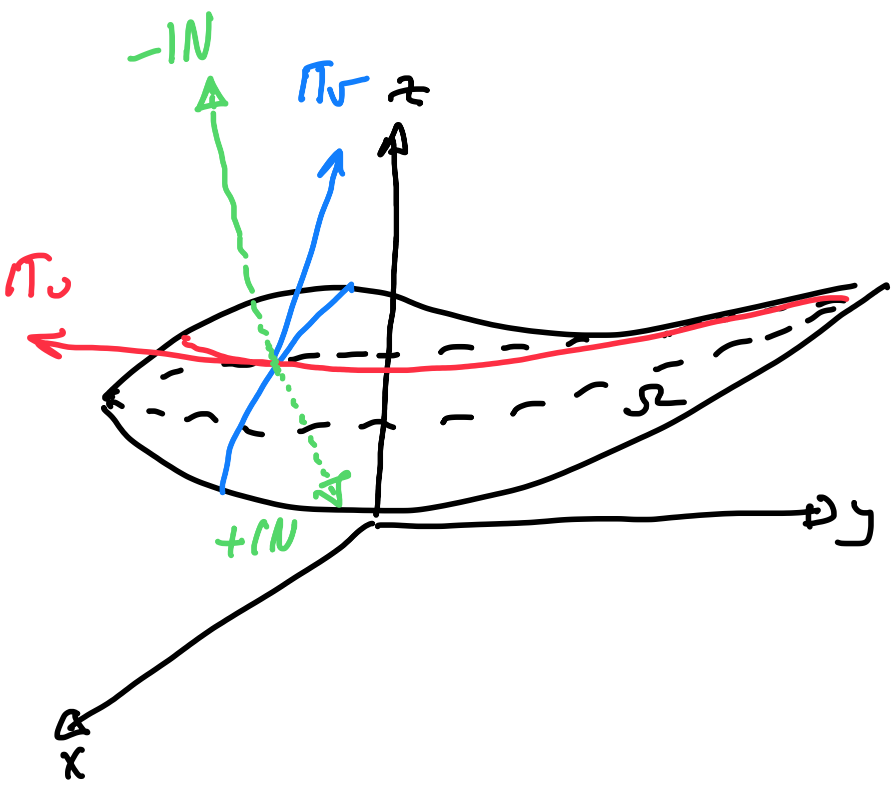
$$= \langle \sqrt{1+v_0^2}\cos u_0,\ \sqrt{1+v_0^2}\ \sin u_0,\ -v_0 \rangle.$$

The equation of the tangent plane at u_0, v_0 is:

$$\mathbb{n}\cdot\langle x(u,v)-x(u_0,v_0),\ y(u,v)-y(u_0,v_0),\ z(u,v)-z(u_0,v_0)\rangle = 0.$$

$$\boxed{\begin{array}{c} x_0(x-x_0)+y_0(y-y_0)-z_0(z-z_0)=0 \\ \text{where } (x_0,y_0,z_0)=\mathbb{r}(u_0,v_0) \end{array}}$$

More on tangent planes and Normal vectors



A surface "Ω" is called smooth if it has a smooth parametrization $\mathbb{r}(u,v)$ such that $x(u,v)$, $y(u,v)$ and $z(u,v)$ are smooth functions, and $\mathbb{r}_u \times \mathbb{r}_v \neq 0$, namely we can find a tangent plane everywhere.

<u>Ex.</u> Find the equation of the tangent plane to the sphere: $x^2+y^2+z^2=9$, at $(\theta, \phi) = (\pi/4, \pi/4)$.

First, let's parametrize the sphere:

$\mathbf{r}(u,v) = \langle 3\sin\phi\cos\theta, 3\sin\phi\sin\theta, 3\cos\phi \rangle.$

We need the plane normal vector, hence..

$\mathbf{T}_\theta(u,v) = \langle -3\sin\phi\sin\theta, 3\sin\phi\cos\theta, 0 \rangle.$

$\mathbf{T}_\phi(u,v) = \langle 3\cos\phi\cos\theta, 3\cos\phi\sin\theta, -3\sin\phi \rangle.$

Hence, $\mathbf{n} = \mathbf{T}_\theta \times \mathbf{T}_\phi$

$= -3\sin\phi \langle 3\sin\phi\cos\theta, 3\sin\phi\sin\theta, 3\cos\theta \rangle.$

$\overset{\text{or}}{=}$ $\mathbf{n} = -3\sin\phi \, \mathbf{r}(u,v).$

$\therefore$ $\mathbf{n}(u_0,v_0) = \langle \frac{-9}{2\sqrt{2}}, \frac{-9}{2\sqrt{2}}, \frac{-9}{2} \rangle$

Hence,

$\mathbf{n} \cdot \langle x-x_0, y-y_0, z-z_0 \rangle$

$= \boxed{\langle \frac{-9}{2\sqrt{2}}, \frac{-9}{2\sqrt{2}}, \frac{-9}{2} \rangle \cdot \langle x-\frac{3}{2}, y-\frac{3}{2}, z-\frac{3}{\sqrt{2}} \rangle = 0}$