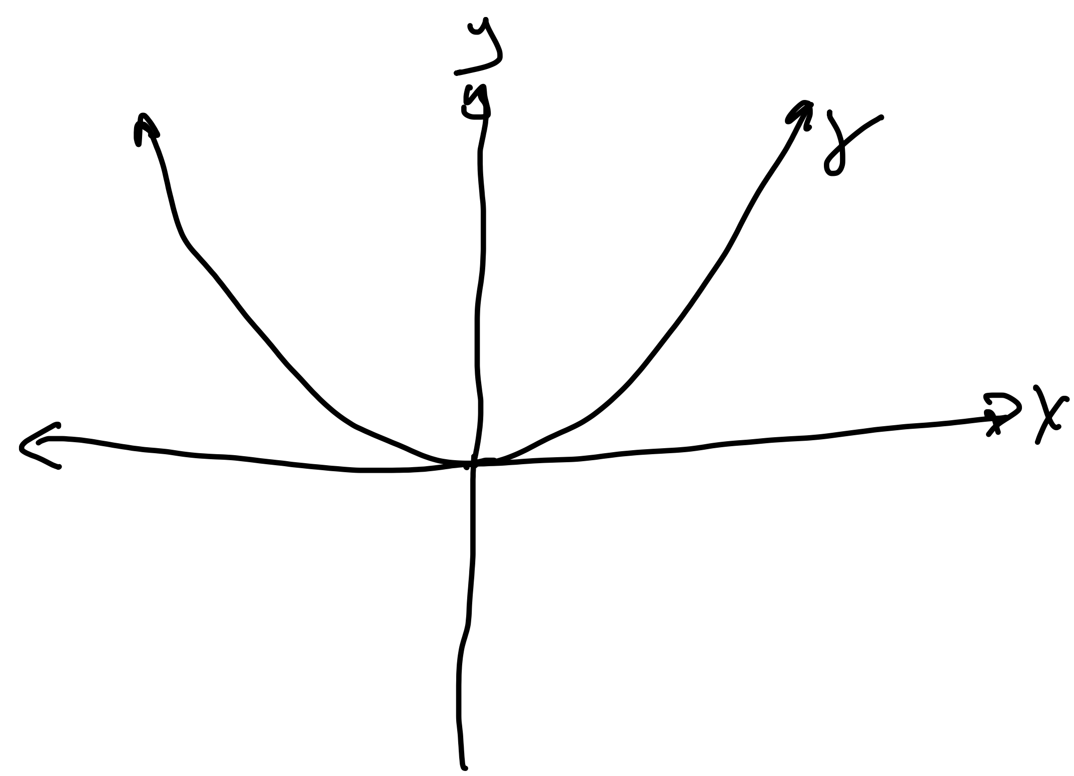
As we showed last class, we can parametrize curves to get a vector valued function that describes every point along the curve in terms of one variable

Eg. $\gamma := \{y = x^2\}$



Let's try and parametrize the curve where $x = t$. With this, we get:

$$\mathbf{r}(t) = \langle t, t^2 \rangle \quad -\infty \leq t \leq \infty$$

→ This is not the only method we can use.

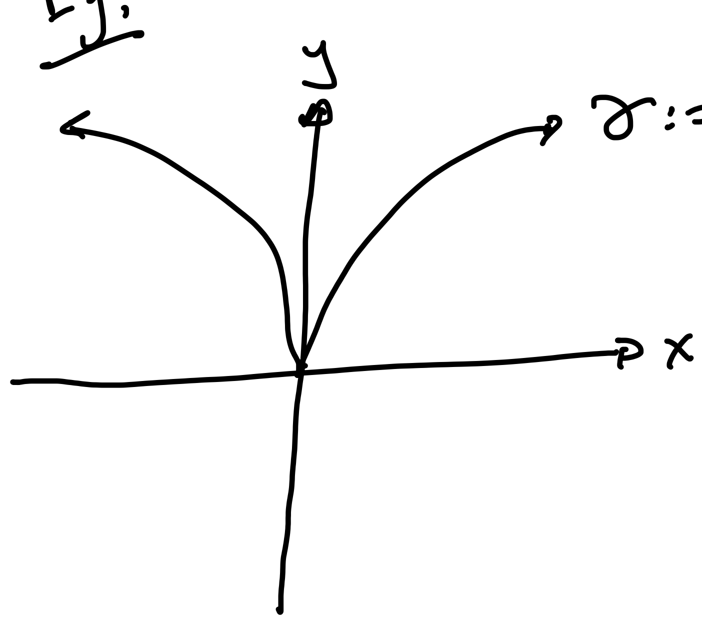
We can also let $y = t$ and hence $t \geq 0$. In this case we just need to break the curve into Left / Right components and add together.

Left ⟹ $x = -\sqrt{t}$

Right ⟹ $x = +\sqrt{t}$

Hence, $\gamma = r(t) = \langle -\sqrt{t}, t \rangle + \langle \sqrt{t}, t \rangle \; ; \; t \geq 0.$

<u>Eg.</u>



$\gamma := \{ y = x^{2/3} \}$

- For this example we need to realize that

$$\left.\frac{dy}{dx}\right|_{x=0} = DNE$$

But $\left.\frac{dx}{dy}\right|_{x=0} = 0.$

Our only option for a real valued and differatiable vector valued function is to let $y=t$ and $x=t^{3/2}$.

With this, we get:

$$r(t) = \langle t^{3/2}, t \rangle, \quad t \geq 0$$

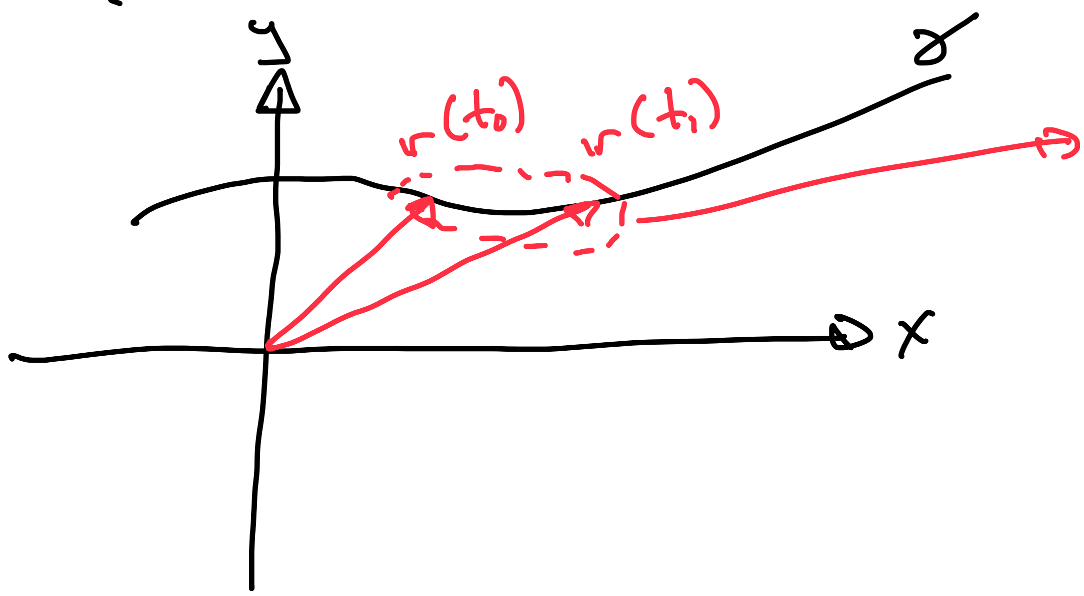
We also see that:

$$r'(t) = \langle \tfrac{3}{2} t^{1/2}, 1 \rangle$$

And

$r'(0) = \langle 0, 1 \rangle$, which is real.

Derivatives.



If we consider $t_1 = t_0 + h$, then:

$$\mathbf{r}'(t) = \frac{d\mathbf{r}(t)}{dt}$$

$$= \lim_{h \to 0} \frac{\mathbf{r}(t_0+h) - \mathbf{r}(t_0)}{h}$$

$$= \langle x'(t), y'(t), z'(t) \rangle.$$

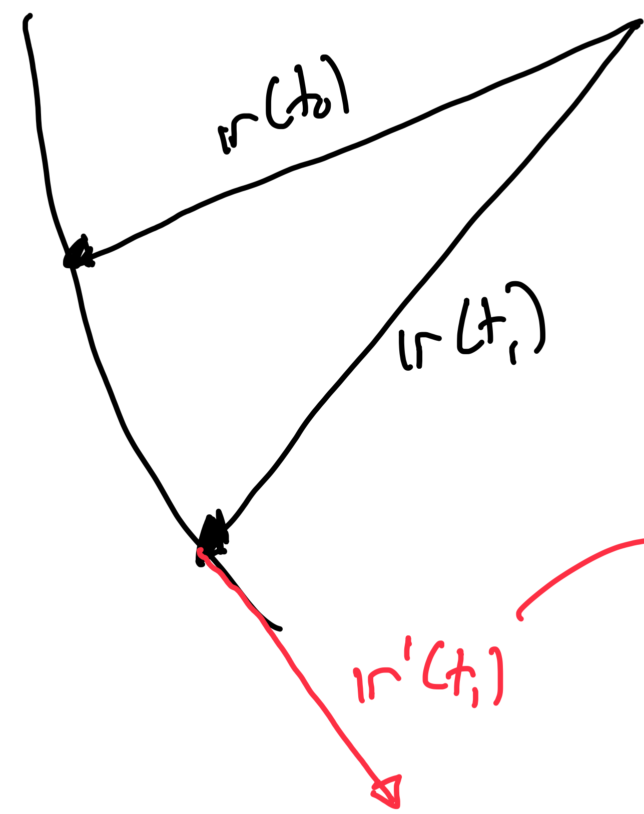
Rules.

1) $$\frac{d}{dt}\left[\vec{a}(t) \cdot \vec{b}(t)\right] = \vec{a}'(t)\,\vec{b}(t) + \vec{a}(t)\,\vec{b}'(t)$$

2) $$\frac{d}{dt}\left[\vec{a}(t) \times \vec{b}(t)\right] = \vec{a}'(t) \times \vec{b}(t) + \vec{a}(t) \times \vec{b}'(t).$$

3) $$\frac{d}{dt}\left[\vec{a}(s(t))\right] = \vec{a}(s(t))\, s'(t).$$

What does the derivative of $\mathbf{r}(t)$ physically represent?



The derivative of $\mathbf{r}(t)$ is the tangent to the curve at "t".

Recall that:

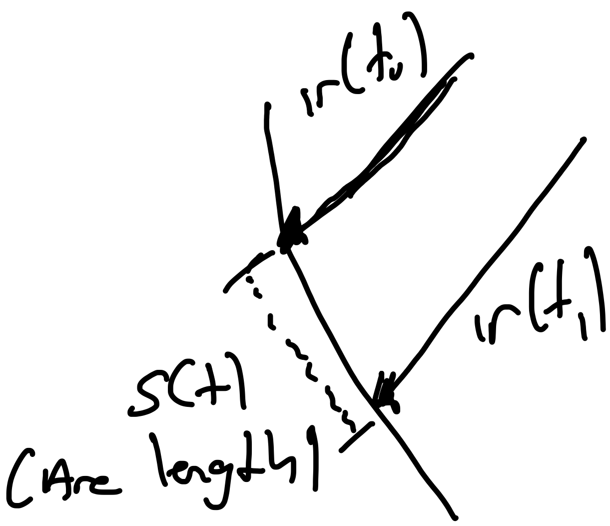
$$\mathbf{r}(t) \cdot \mathbf{r}'(t) = 0$$

(orthogonal vectors).

- Hence, the unit tangent vector at any point "t" on the curve is:

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}$$

We also see that the "arc length" is related to the magnitude of the local "velocity" vector, ie. the speed, by:



$$\frac{ds}{dt} = \left| \frac{dr}{dt} \right|$$

<u>Ans</u>

$$S(T) = \int_{T_0}^{T} \left| \frac{dr}{dt} \right| dt + S(T_0)$$

Definitions:

Think of these in terms of the pathway a ball travels after it was thrown, ie. t= time.

Position $\Rightarrow$ $\mathbb{r}(t) = \langle x(t), y(t), z(t) \rangle.$

Velocity $\Rightarrow$ $\mathbb{r}'(t) = \langle x'(t), y'(t), z'(t) \rangle.$

Speed $\Rightarrow$ $|\mathbb{r}'(t)| = \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2}$

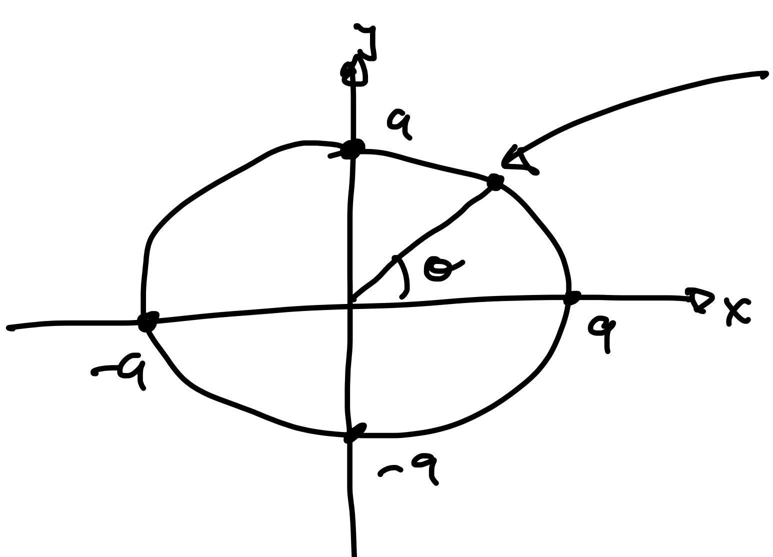
$= \frac{ds}{dt}$

Acceleration $\Rightarrow$ $\mathbb{r}''(t) = \langle x''(t), y''(t), z''(t) \rangle.$

Distance travelled $\Rightarrow$ $S(T) - S(T_0) = \int_{T_0}^{T} |\mathbb{r}'(t)|\, dt.$

Ex,

What is the arc length of a circle of radius a?



$x = a\cos\theta$
$y = a\sin\theta$

→ let's parametrize the curve with "θ"

$$\mathbf{r}(\theta) = \langle a\cos\theta, a\sin\theta \rangle \quad , \quad 0 \leq \theta \leq 2\pi$$

$$\mathbf{r}'(\theta) = \langle -a\sin\theta , a\cos\theta \rangle$$

Hence, $$|\mathbf{r}'(\theta)| = a$$

Now, we can solve for the arc length.

$$S(2\pi) - S(0) = \int_0^{2\pi} a \, dt$$

$$\boxed{S(2\pi) = 2\pi a}$$