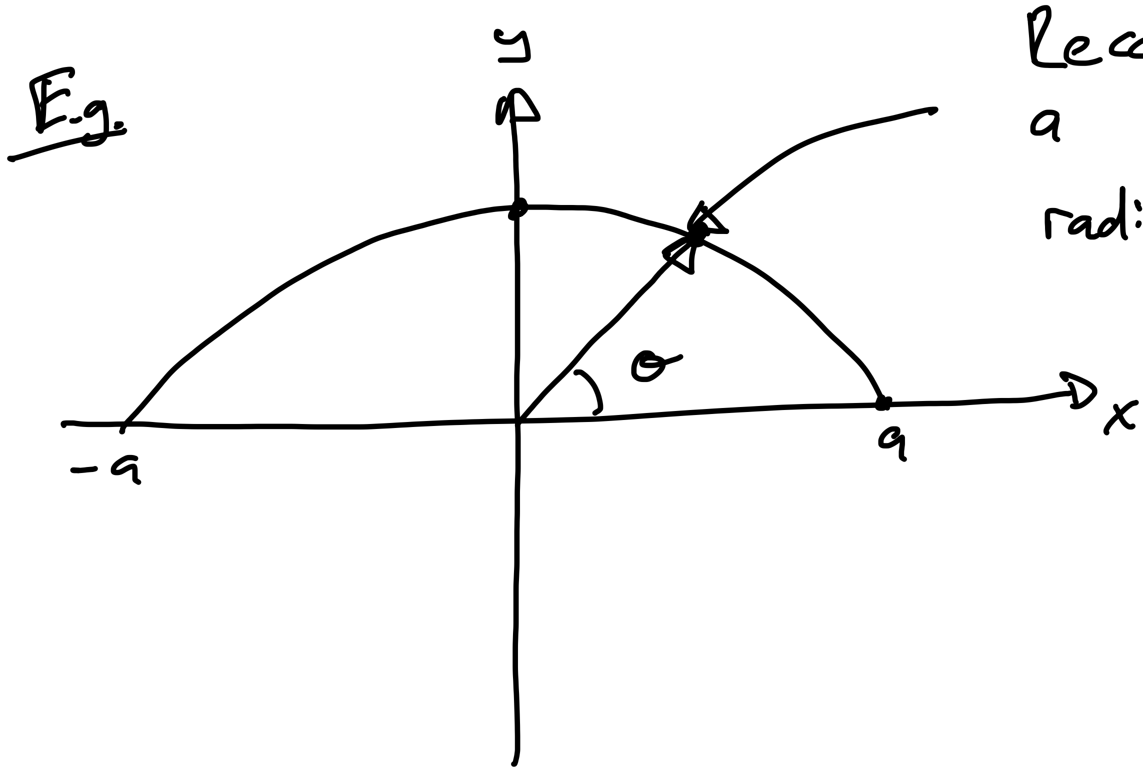
Parametrizations

E.g.



Recall that for a half circle of radius a:

$$\boxed{\begin{aligned} x^2+y^2 &= a^2 \\ x &= a\cos\theta \\ y &= a\sin\theta \\ y &\geq 0 \end{aligned}}$$

Method #1

- Parametrize in polar coordinates.

$$\mathbf{r}(\theta) = \langle a\cos\theta, a\sin\theta \rangle, \quad 0 \leq \theta \leq \pi$$

Method #2.

Parametrize in cartesian coordinates.

$$a^2 = x^2+y^2 \Rightarrow y(x) = \sqrt{a^2-x^2}$$

$\therefore$ For $t=x$, we are left with

$$\mathbf{r}(t) = \langle t, \sqrt{a^2-t^2} \rangle, \quad -a \leq t \leq a$$

Method #3

- Parametrize with "arc length"

$||r'(\theta)|| = \langle -a\sin\theta, a\cos\theta \rangle.$

Hence, $s(\theta) = a\theta$ OR $\theta = s/a$

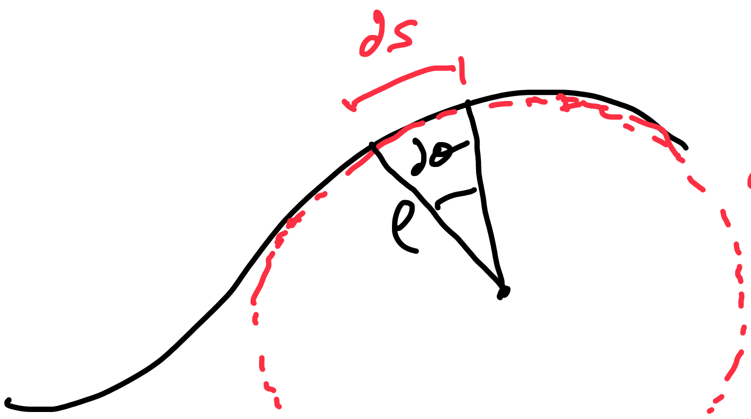
Revisiting Method #1, we can substitute this in and yield our 3rd parametrization.

$$r\left(\frac{s}{a}\right) = \left\langle a\cos\left(\frac{s}{a}\right), a\sin\left(s/a\right) \right\rangle, \quad 0 \leq s \leq a\pi$$

Curvature

Definitions : 1) 'ρ' is called the radius of curvature.

2) $k = \frac{1}{\rho}$ (kappa) is the curvature and is a measure of how tight the curve turns

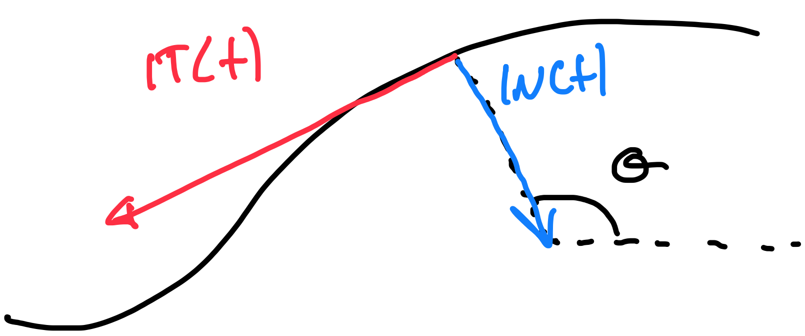


Unit circle.

we can see by trigonometry that $\rho = \left| \frac{ds}{d\theta} \right|$, and

$$k = \left| \frac{ds}{d\theta} \right|^{-1}$$

- Let's extend this analysis to find the unit tangent & unit normal vectors along a given curve.



we have already found that the unit tangent vector is: $|T(t)| = \frac{|r'(t)|}{||r'(t)||}$

The unit normal vector is:

$$|N(t)| = \frac{|T'(t)|}{||T'(t)||}, \text{ and}$$

points towards the center of curvature.

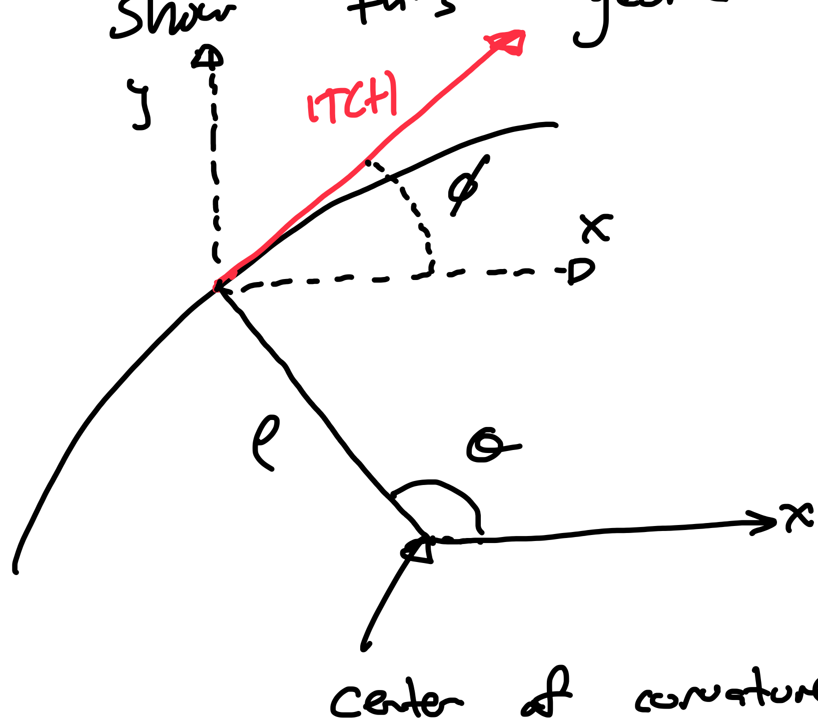
→ Let's extend this analysis and find the unit normal & unit tangent vectors in terms of "arc length"

We already know that: ① $\frac{ds}{dt} = \left| \mathbf{r}'(t) \right|$

② $\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}$

Hence, $\mathbf{T}(s) = \frac{\frac{d\mathbf{r}}{dt}}{\frac{ds}{dt}} = \frac{d\mathbf{r}}{ds}$

Let's show this geometrically.



• We already know that

$\rho = \left| \frac{ds}{d\theta} \right|$, but we also see that $\phi = \theta \pm c$, where c is a constant.

Hence, $\rho = \left| \frac{ds}{\partial \phi} \right|$; $k = \left| \frac{ds}{d\phi} \right|^{-1}$ as well.

With this, we can show that:

$$\frac{d\mathbb{T}}{ds} = \frac{d\mathbb{T}}{d\phi}\frac{d\phi}{ds}$$

$$= \mathbb{N}(s)\, k(s)$$

- A more convenient way to solve for the unit normal vector in relation to arc length is

$$\boxed{\mathbb{N}(s) = \frac{d\mathbb{T}}{ds} \Big/ k(s)}$$

<u>Parametrizations with general 't'</u>

Moving from arclength to 't' is easy.....

$$\frac{d\mathbb{T}}{dt} = \frac{d\mathbb{T}}{ds}\frac{ds}{dt} = k(t)\mathbb{N}(t)\frac{ds}{dt}$$

→ Recall that $s = ct$, where c is a constant.

Since, $k(t) = \left| \frac{d\mathbf{T}(t)}{dt} \right| / ds/dt$, we arrive at

$\mathbf{N}(t) = \frac{d\mathbf{T}}{dt} \Big/ \left| \frac{d\mathbf{T}(t)}{dt} \right|$, which agrees with what we had before.

Ex.

Show that for a circle of radius a, the curvature is $k = \frac{1}{a}$.

- We know that for a circle: $\boxed{\begin{array}{l} x = a\cos t \\ y = a\sin t \\ t = \theta \end{array}}$

we can write our polar parametrization as,

$\mathbf{r}(t) = \langle a\cos t, a\sin t \rangle$, $0 \le t \le 2\pi$

Now, the curvature is $k(t) = \left| \frac{d\mathbf{T}}{dt} \right| / ds/dt$,

So let's find $\mathbf{T}(t)$ first.

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = \frac{\langle -\cancel{a}\sin t, \cancel{a}\cos t \rangle}{\cancel{a}}$$

<u>Ans</u>

$$\mathbf{T}'(t) = \langle -a\cos t, -a\sin t \rangle$$

Now, we can find $ds/dt = \left|\frac{d\mathbf{r}}{dt}\right| = a.$

Hence,

$$\boxed{k(t) = \frac{|\mathbf{T}'(t)|}{a} = \frac{1}{a}}$$