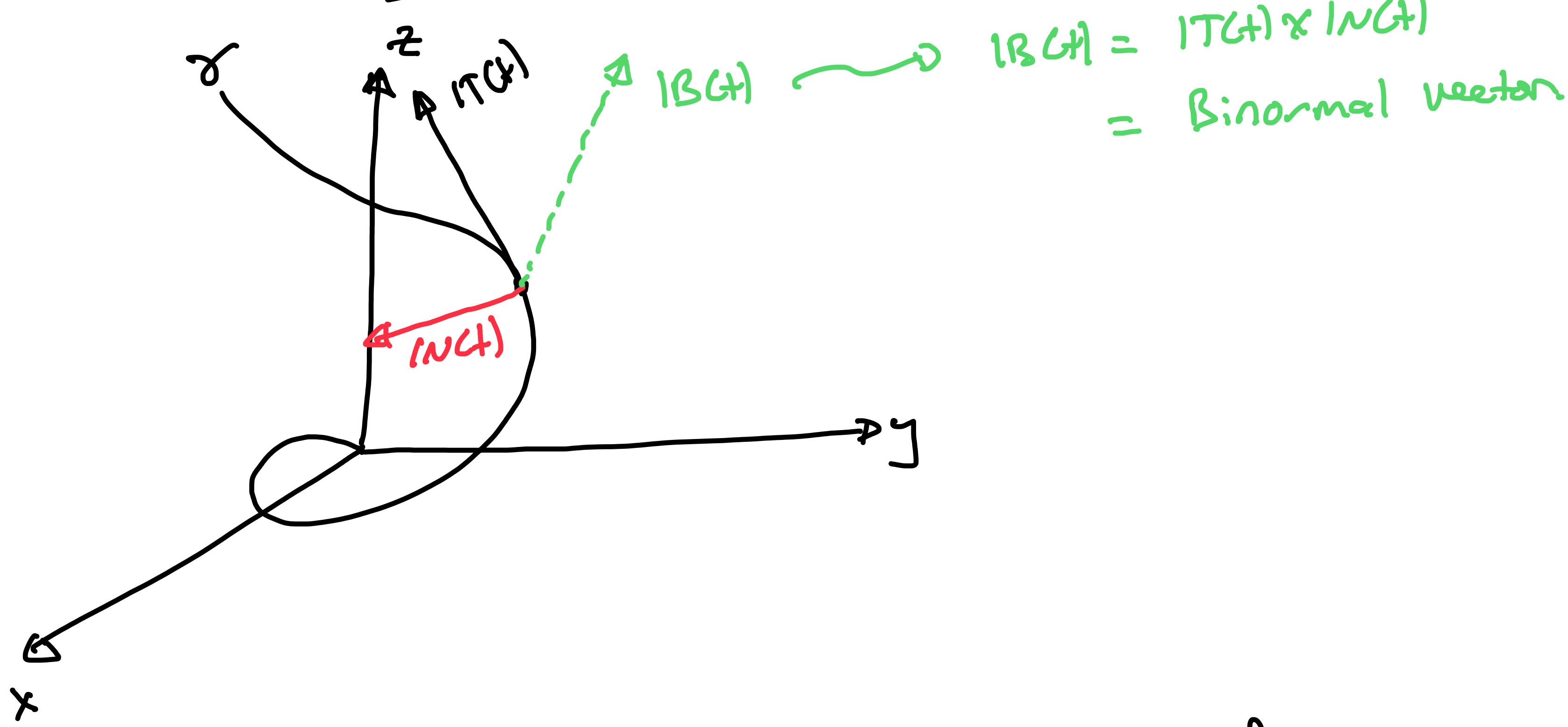
# Curves in 3D.




→ So far we've developed expressions for the unit tangent / normal vector along a curve. Our objective here is to do the same, but now in 3D. This gives us a moving frame of reference as you march along the curve, i.e. "Frenet Frame".

1) Recall that the Binormal vector is orthogonal to both T(t) and N(t).

$\therefore$ IF $B(t) = T(t) \times N(t)$, then

$$B(s) = T(s) \times N(s)$$

→ We can determine how the binormal vector moves along the curve by differentiating with respect to 's'..

Hence,

$$\frac{d\mathbb{B}}{ds} = \frac{d}{ds}\left[\mathbb{T}(s) \times \mathbb{N}(s)\right]$$

$$= \frac{d\mathbb{T}}{ds} \times \mathbb{N}(s) + \mathbb{T}(s) \times \frac{d\mathbb{N}(s)}{ds}$$

→ This is essentially a measure of how the plane twists as you move along the curve.

Since, $\frac{d\mathbb{T}}{ds}$ is parallel to $\mathbb{N}(s)$, ie. $\frac{d\mathbb{T}}{ds} \times N(s) = 0$,

We arrive at :

$$\boxed{\frac{d\mathbb{B}}{ds} = \mathbb{T}(s) \times \frac{d\mathbb{N}(s)}{ds}}$$

- To make our lives easier, let's define a scalar variable that describes the frames twisting motion.

i.e. $\frac{d\mathbb{B}}{ds} = -\tau(s)\,\mathbb{N}(s)$, where $\tau(s)$ = Torsion.

Note that for $\tau(s) > 0$, the rotation is counter-clockwise (i.e. right-hand rule).

→ We've already found how the tangent vector changes along the curve, i.e.

$$\frac{d\mathbb{T}}{ds} = k(s)\,\mathbb{N}(s).$$

but how does the normal vector change?

Simple,

$$\frac{d\mathbb{N}(s)}{ds} = \frac{d}{ds}\left[\mathbb{B}(s) \times \mathbb{T}(s)\right]$$

$$= \frac{d\mathbb{B}(s)}{ds} \times \mathbb{T}(s) + \mathbb{B}(s) \times \frac{d\mathbb{T}(s)}{ds}$$

$$= \left[-\tau(s)\mathbb{N}(s) \times \mathbb{T}(s)\right] + \left[\mathbb{B}(s) \times k(s)\mathbb{N}(s)\right]$$

Recall that: $\mathbb{B}(s) = \mathbb{T}(s) \times \mathbb{N}(s)$, so

$-\mathbb{B}(s) = \mathbb{N}(s) \times \mathbb{T}(s)$ AND

$-\mathbb{T}(s) = \mathbb{B}(s) \times \mathbb{N}(s)$

$\boxed{\text{VALIDATE WITH R.H.R}}$

Now, with these expressions we arrive at:

$$\boxed{\frac{d\mathbb{N}(s)}{ds} = \tau(s)\,\mathbb{B}(s) - k(s)\,\mathbb{T}(s)}$$

- With this last expression, we're left with a system of ODE's that define the frame along a given curve. It has a unique solution on the curve in if we specify a point on the space

$$\frac{d}{ds}\begin{bmatrix} \mathbb{T} \\ \mathbb{N} \\ \mathbb{B} \end{bmatrix} = \begin{bmatrix} 0 & k(s) & 0 \\ -k(s) & 0 & \tau(s) \\ 0 & -\tau(s) & 0 \end{bmatrix} \begin{bmatrix} \mathbb{T}(s) \\ \mathbb{N}(s) \\ \mathbb{B}(s) \end{bmatrix}$$

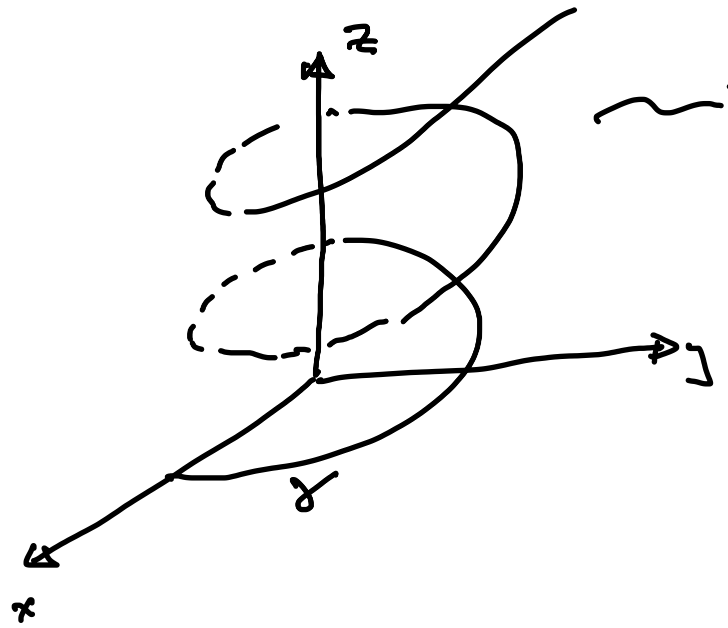
Notes

$$\frac{d\mathbb{B}(s)}{ds} \cdot \mathbb{N}(s) = -\tau(s)\,\mathbb{N}(s) \cdot \mathbb{N}(s)$$

$$= -\tau(s)$$

Here, $\boxed{\tau(s) = -\frac{d\mathbb{B}(s)}{ds} \cdot \mathbb{N}(s)}$

Ex, Find the torsion of the helix.




$\mathbf{r}(t) = \langle a\cos t, a\sin t, bt \rangle$.

- For practise, let's parametrize the helix in terms of arclength and solve for $\tau(s)$.

Step # 1

$\mathbf{r}'(t) = \langle -a\sin t, a\cos t, b \rangle$.

$|\mathbf{r}'(t)| = \sqrt{a^2+b^2}$

Ans

$$s(t) = \int_0^T |\mathbf{r}'(u)| \, du = t\underbrace{\sqrt{a^2+b^2}}$$

Let $c = \sqrt{a^2+b^2}$

Hence,

$$\mathbf{r}(s) = \langle a\cos(s/c), a\sin(s/c), \tfrac{bs}{c} \rangle$$

Step #2.

Construct $\{ \mathbb{T}(s), \mathbb{B}(s), \mathbb{N}(s) \}$ Frame.

$$\mathbb{T}(s) = \frac{d\mathbb{r}}{ds} = \langle -\frac{a}{c}\sin\left(\frac{s}{c}\right), \frac{a}{c}\cos\left(\frac{s}{c}\right), b/c \rangle.$$

$$\mathbb{N}(s) = \frac{d\mathbb{T}}{ds} \Big/ \left|\frac{d\mathbb{T}}{ds}\right| = \langle -\cos(s/c), -\sin(s/c), 0 \rangle.$$

$$\mathbb{B}(s) = \mathbb{T}(s) \times \mathbb{N}(s)$$

$$= \frac{1}{c} \langle b\sin(s/c), -b\cos(s/c), a \rangle.$$

Step #3.

Solve for torsion.

$$\tau(s) = -\frac{d\mathbb{B}}{ds} \cdot \mathbb{N}(s)$$

$$\frac{d\mathbb{B}}{ds} = \frac{1}{c^2} \langle b\cos(s/c), b\sin(s/c), 0 \rangle.$$

Hence, $\tau(s) = -\frac{1}{c^2} \langle b\cos(s/c), b\sin(s/c), 0 \rangle \cdot$

$\langle -\cos(s/c), -\sin(s/c), 0 \rangle.$

$$= \frac{b}{c^2} = \boxed{\frac{b}{a^2+b^2}}$$

Note that the curvature is just:

$$\kappa(s) = \left| \frac{dT}{ds} \right| = \frac{a}{a^2+b^2}$$

I will do this same example in the whiteboard session but w.r.t parametrization with 't'.