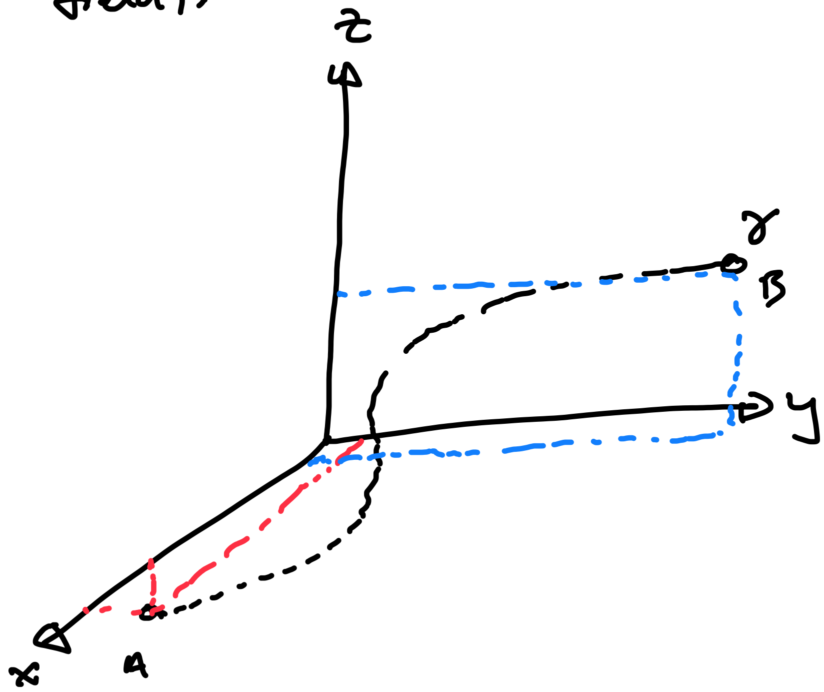
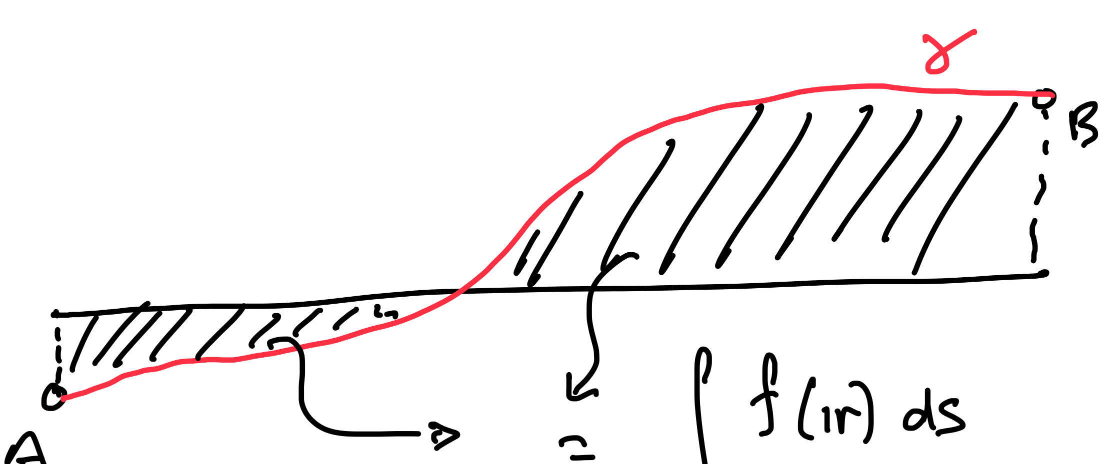
Path integrals.

- Think of this as a measure of "work" done on a particle moving along a curve γ inside a scalar force field $f(x,y,z)$ (e.g. a pressure field).



→ If we project this pathway from [A, B] to a scalar variable 't' (parametrization variable) we get,



$$= \int_{\gamma} f(\mathbf{r})\, ds$$

Hence,
$$\int_{\gamma} f(x,y,z)\,ds = \int_a^b f(\mathbf{r}(t))\,\underbrace{|\mathbf{r}'(t)|}\,dt.$$

Recall that:

$$\frac{ds}{dt} = |\mathbf{r}'(t)|$$

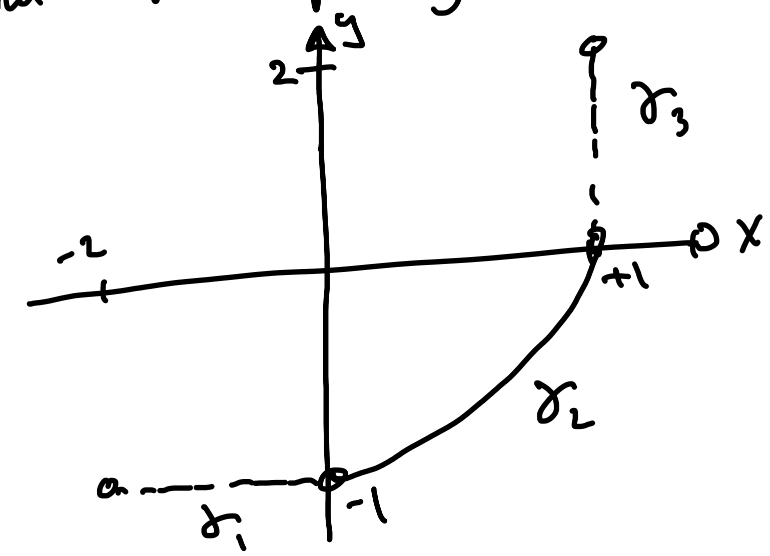
or

$$ds = |\mathbf{r}'(t)|\,dt.$$

E.g.

Let the scalar force field be : $f(x,y) = 4x_1^3$

and the pathway :



γ_1 : $x = t,\ y = -1$; $-2 \le t \le 0$

γ_2 : $x = t,\ y = t^3 - 1$; $0 \le t \le 1$

γ_3 : $x = 1,\ y = t$; $0 \le t \le 2$.

$$\therefore \int_{\gamma} f(x,y,z)\,ds = \int_{\gamma_1} f_1(x,y,z)\,ds_1 + \int_{\gamma_2} f_2(x,y,z)\,ds_2 + \int_{\gamma_3} f_3(x,y,z)\,ds_3$$

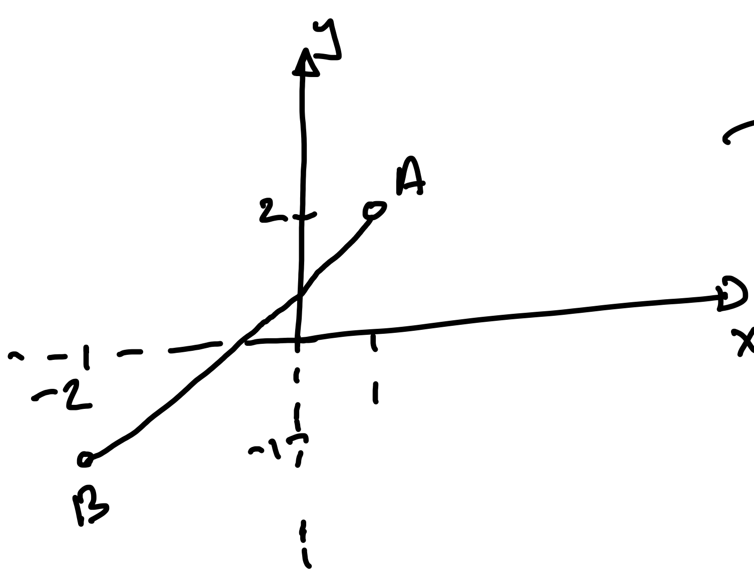
$$= \int_{-2}^{0} 4t^3\sqrt{1^2+0^2}\,dt + \int_0^1 4t^3\sqrt{1^2+(3t^2)^2}\,dt + \int_0^2 4(1)^3\sqrt{0^2+1^2}\,dt$$

$$= -16 - 2.27 + 8 = \underline{-5.732}$$

Ex,

Evaluate: $\displaystyle\int_\gamma 4x^3\,ds$ for a line segment

from $\underset{[A]}{(1,2)}$ to $\underset{[B]}{(-2,-1)}$.



$$\rightsquigarrow \; r(t) = \langle 1 + c_1 t,\ 2 + c_2 t \rangle.$$

$$c_1 = -2 - 1 = -3$$

$$c_2 = -1 - 2 = -3.$$

$$r(t) = \langle 1 - 3t,\ 2 - 3t \rangle; \quad 0 \le t \le 1$$

Hence, $$\int_\gamma 4x^3\,ds = \int_0^1 4(1-3t)^3 \sqrt{9+9}\,dt$$

$$= \underline{-15\sqrt{2}.}$$

Ex.

$$\int_\gamma xyz\,ds, \quad \text{where } \mathbf{r}(t) = \langle \cos t, \sin t, 3t \rangle; \quad 0 \le t \le 4\pi$$

$$\int_\gamma xyz\,ds = \int_0^{4\pi} \cos t \sin t \, 3t \sqrt{10}\, dt$$

Recall that: $\cos t \sin t = \frac{1}{2} \sin 2t$

Hence,

$$\frac{3\sqrt{10}}{2} \int_0^{4\pi} t \sin 2t \, dt = \frac{3\sqrt{10}}{2} \left[-\frac{1}{2} t \cos(2t) \Big|_0^{4\pi} + \frac{1}{2} \int_0^{4\pi} \cos(2t)\, dt \right]$$

$$= \frac{3\sqrt{10}}{2} \left[-2\pi \right] = \boxed{-3\pi\sqrt{10}}$$