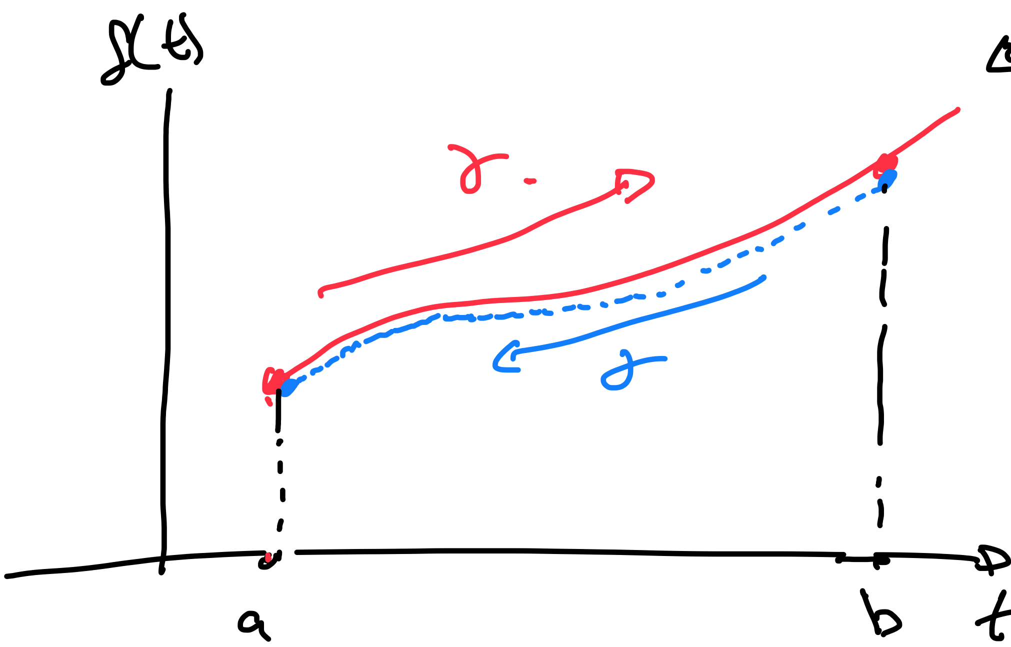
Clarification from yesterday's notes.



Here, the curves are overlapping.

- In general, a given parametrization $x = x(t), y = y(t)$, $a \leq t \leq b$, determines an "orientation" of the curve γ, with a positive orientation corresponding to INCREASING values of 't'. With this, we have:

$$-\int_{\gamma} f(\mathbf{r}(t))\,|\mathbf{r}'(t)|\,dt = \int_{\gamma} f(\mathbf{r}(t))\,|\mathbf{r}'(t)|\,dt.$$

OR

$$-\int_{b}^{a} f(\mathbf{r}(t))\,|\mathbf{r}'(t)|\,dt = \int_{a}^{b} f(\mathbf{r}(t))\,|\mathbf{r}'(t)|\,dt.$$

• Note! that this does not effect the integration with respect to "arc-length",

ie.

$$\int_{\gamma} f(x,y)\, ds = \int_{\gamma} f(x,y)\, ds .$$

This is because ds is always positive, whereas $\Delta x(t)$ and $\Delta y(t)$ change sign when we reverse the direction.... hence, be careful when deriving your parametrized vector function to ensure that $a \le t \le b$ and the curve is "positively" oriented.

Vector fields

- Typically, two types of vector fields arrise in physics / mathematics :

  ① Velocity field ($\mathbb{v}(x,y,z)$)

  ② Force field. ($\mathbb{F}(x,y,z)$)

→ Imagine you throw a ball in a moving river, the velocity field as the moving river results in viscous forces and inertial forces acting on the ball. But, in addition to those, gravitational and buoyant forces are also acting on the ball.

In other words, in reality, velocity fields & force fields are closely linked... i.e., force fields can result in motion, and velocity fields can result in a force(s).

→ The field of fluid mechanics studies fluid motion and the forces that either cause, or result in fluids moving... This area of work is a personal favorite.

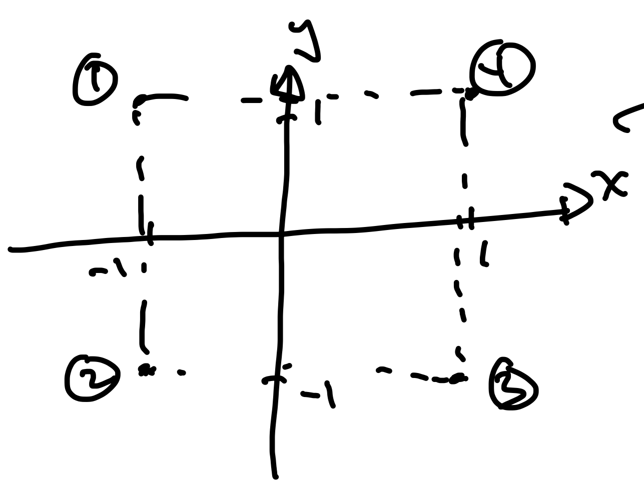
For argument's sake, let's first look at velocity fields, where we assign a velocity vector to every point in space,

$$\mathbf{v}(x,y,z) = \langle v_x(x,y,z), v_y(x,y,z), v_z(x,y,z) \rangle.$$

Eg.

$$\mathbf{v}(x,y) = \langle v_x, v_y \rangle = \langle -y, x \rangle$$

Q. Is the fluid rotating?



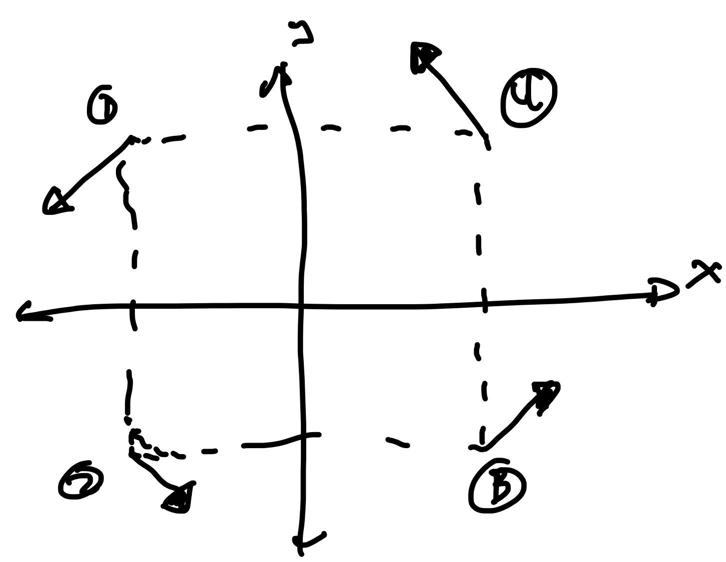
① $\mathbf{v}(-1,1) = \langle -1,-1 \rangle$

② $\mathbf{v}(-1,-1) = \langle 1,-1 \rangle$

③ $\mathbf{v}(1,-1) = \langle 1,1 \rangle$

④ $\mathbf{v}(1,1) = \langle -1,1 \rangle$

Let's draw out the vectors:



Yes, the fluid is rotating.

Gradients

Potential functions: A vector field, whether it be a velocity / force field, is said to be conservative if there exists a scalar and continuous function ϕ such that:

$$\mathbb{V} = \nabla \phi \quad \text{(velocity field)}$$

$$\underline{\underline{\text{OR}}}$$

$$\mathbb{F} = \nabla \phi \quad \text{(Force field)}.$$

Note

The del operator, $\nabla = \frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}$.

- This is a common method to use in many physical problems (including fluid mechanics) where viscous diffusion can be neglected.

Ex.

Let's say we have a force field / velocity field:

$$\mathbb{F}(x,y) = \langle x, -y \rangle \ldots$$ Is this conservative?

We need a function that satisfies:

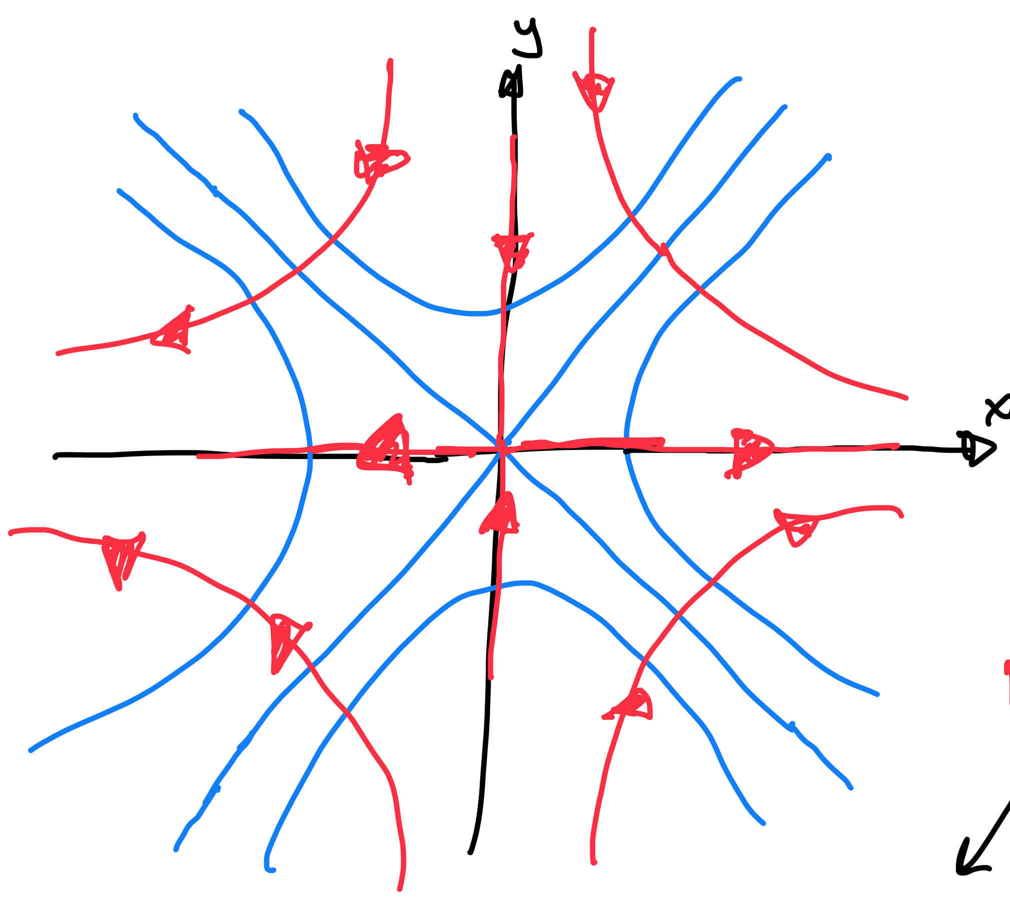
$$\nabla \phi = \langle x, -y \rangle.$$

① $\frac{\partial \phi}{\partial x} = x \Rightarrow \phi(x,y) = \frac{x^2}{2} + f(y)$

② $\frac{\partial \phi}{\partial y} = -y \Rightarrow \phi(x,y) = \frac{-y^2}{2} + f(x).$

Hence, $f(x) = \frac{x^2}{2}$; $f(y) = \frac{-y^2}{2}.$

$\therefore \phi(x,y) = \frac{x^2}{2} - \frac{y^2}{2}$ Ans is certainly conservative.



Blue: $\phi(x,y)$

Red: $\mathbb{F}(x,y)$.

Hence, $\phi(x,y)$ is orthogonal to $\mathbb{F}(x,y)$.

Home

- If you're interested, check out the classic "flow around a cylinder" solution to see the potential function usefulness.

Irrotational flow.

- Curls of vector fields are important in physics and describe the rotation of a vector field.
- They also help distinguish if vector fields are conservative.

Eg.

$$\mathbb{F}(x,y,z) = \nabla\phi$$

$$\text{curl of } \mathbb{F}(x,y,z) = \nabla\times\nabla\phi$$

$$\nabla\times\nabla\phi = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial\phi}{\partial x} & \frac{\partial\phi}{\partial y} & \frac{\partial\phi}{\partial z} \end{vmatrix} = 0.$$

Hence, conservative vector functions are irrotational!

A simple and FAST way to check if a vector field is conservative is to calculate the curl... if it's zero, then it's conservative.

Streamlines:

→ Streamlines map out trajectories of massless particles in a vector field.

Formal definition ⇒ $\boxed{\mathbf{r}'(t) \times \mathbf{v}(\mathbf{r}(t)) = 0.}$

- This gives a family of curves that are instantaneously tangent to the velocity field.

→ This really comes down to defining the velocity field as:

$$\mathbf{v}(x, y, z) = \nabla \times \underset{\sim}{\vec{\psi}}$$

ψ is the stream function. (vector potential).

- This is analagous to the potential function (velocity potential), but is $\underline{\text{not}}$ limited to irrotational flows. The only condition that needs to be met is that the velocity field must be divergence free,

$$\boxed{\nabla \cdot \mathbf{v} = 0}$$

Ex.

Find the Stream function for: $v(x,y) = \langle -y, x \rangle$.

$\Rightarrow$ we Seek to find a function such that,

$$v = \nabla \times \vec{\psi}$$

we can already see that $\vec{\psi} = \langle 0, 0, \psi \rangle$, for this to work...

Hence,

$$\nabla \times \vec{\psi} = \left\langle \frac{\partial \psi}{\partial y}, -\frac{\partial \psi}{\partial x}, 0 \right\rangle$$

$\therefore$ (1) $\dfrac{\partial \psi}{\partial y} = -y \Rightarrow \psi(x,y) = \dfrac{-y^2}{2} + f(x)$

(2) $-\dfrac{\partial \psi}{\partial x} = x \Rightarrow \psi(x,y) = -\dfrac{x^2}{2} + f(y)$.

$$\therefore \boxed{\psi(x,y) = \frac{x^2}{2} + \frac{y^2}{2} \quad \text{AND} \quad \psi(x,y) \geq 0}$$

Recall that $\psi(x,y)$ is a constant, and a constant multiplied by a constant is just another constant

We can get that same result by:

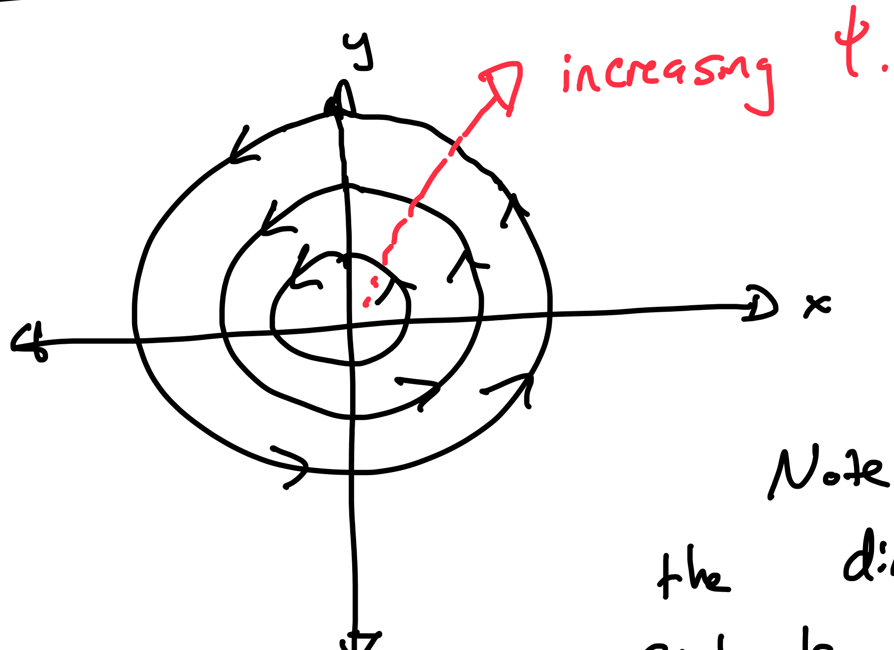
$$\mathbf{r}'(t) \times \mathbf{v}(\mathbf{r}(t)) = 0.$$

$$\left[\frac{\partial x}{\partial t}, \frac{\partial y}{\partial t}, 0\right] \times \left[v_x(t), v_y(t), 0\right] = 0$$

$$x\frac{dx}{dt} + y\frac{dy}{dt} = 0$$

AND

$$\boxed{\frac{x^2}{2} + \frac{y^2}{2} = C = \psi}$$



Note that to get the direction of rotation, evaluate $\mathbf{v}(x,y)$.